Engineering Mathematics Ⅱ Hw#4 日河工 2B

20 April 2005

1) Let $f(x) = xe^{-|x|}$ (Example 14.2)

(a)Show that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges

(b)Sketch $f(x) = xe^{-|x|}$ and judge whether $f(x)$ is piecewise smooth or not

(c)Write the Fourier integral of $f(x)$ on the real line

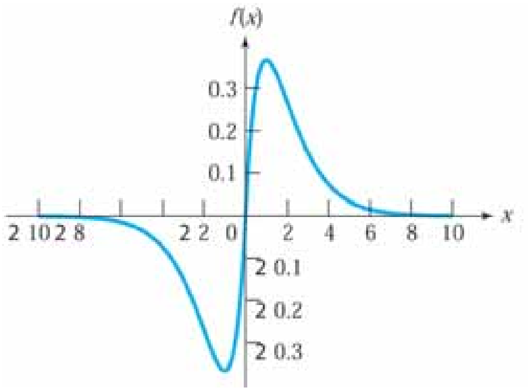
(a)

$$\int_{-\infty}^{\infty} \left|xe^{-|x|}\right| dx = 2\int_0^{\infty} xe^{-x} dx$$
$$= 2\left\{-xe^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x} dx\right\}$$
$$= 2\left\{-e^{-x}\right\}_0^{\infty}$$
$$= 2\left\{-e^{-\infty} + e^{0}\right\}$$
$$= 2$$

===➔ $f(x) = xe^{-|x|}$ is said to be absolutely integrable (and then integrable) on $[-\infty, \infty]$.

(a)The graph of $f(x) = xe^{-|x|}$

$f(x)$ is piecewise smooth



(c)

$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi$$

$$B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi$$

$$A_\omega + iB_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi + i\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)[\cos(\omega\xi) + i\sin(\omega\xi)]d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)e^{i\omega\xi}d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \xi e^{-|\xi|}e^{i\omega\xi}d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \xi e^{-|\xi|+i\omega\xi}d\xi$$

$$= \frac{1}{\pi}\left\{\int_{-\infty}^{0} \xi e^{\xi+i\omega\xi}d\xi + \int_{0}^{\infty} \xi e^{-\xi+i\omega\xi}d\xi\right\}$$

$$A_\omega + iB_\omega = \frac{1}{\pi}\left\{\int_{-\infty}^{0} \xi e^{\xi+i\omega\xi}d\xi + \int_{0}^{\infty} \xi e^{-\xi+i\omega\xi}d\xi\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\xi}{1+i\omega}e^{\xi+i\omega\xi}\Big|_{-\infty}^{0} - \frac{1}{1+i\omega}\int_{-\infty}^{0} e^{\xi+i\omega\xi}\right\} +$$

$$\frac{1}{\pi}\left\{\frac{\xi}{-1+i\omega}e^{-\xi+i\omega\xi}\Big|_{0}^{\infty} - \frac{1}{-1+i\omega}\int_{0}^{\infty} e^{-\xi+i\omega\xi}\right\}$$

➔

$$= \frac{1}{\pi}\left\{-\frac{1}{(1+i\omega)^2}e^{\xi+i\omega\xi}\Big|_{-\infty}^{0} - \frac{1}{(-1+i\omega)^2}e^{-\xi+i\omega\xi}\Big|_{0}^{\infty}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{(1+i\omega)^2} + \frac{1}{(-1+i\omega)^2}\right\}$$

$$= \frac{4i\omega}{(1+\omega^2)^2}$$

➔

$$A_\omega = 0$$

$$B_\omega = \frac{4\omega}{(1+\omega^2)^2}$$

Note: you should compare the problem to Section 14.3 Problem 1. (HW5)

2) Section 14.1 Problems 1.

$$f(x) = \begin{cases} x & for \quad -\pi \le x \le \pi \\ 0 & for \quad |x| > \pi \end{cases}$$

$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} \xi\cos(\omega\xi)d\xi = 0 \quad \text{(note } \xi\cos(\omega\xi) \text{ is odd)}$$

$$B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \xi\sin(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\left\{-\xi\frac{\cos(\omega\xi)}{\omega}\Bigg|_0^\pi + \int_0^\pi \frac{\cos(\omega\xi)}{\omega}d\xi\right\} = \frac{2}{\pi}\left\{-\xi\frac{\cos(\omega\xi)}{\omega}\Bigg|_0^\pi + \frac{\sin(\omega\xi)}{\omega^2}\Bigg|_0^\pi\right\}$$

$$= 2\left[\frac{\sin(\omega\pi)}{\pi\omega^2} - \frac{1}{\omega}\cos(\omega\pi)\right]$$

➔The Fourier integral of f is

$$\int_0^\infty \left[\frac{2\sin(\omega\pi)}{\pi\omega^2} - \frac{2\cos(\omega\pi)}{\omega}\right]\sin(\omega x)d\omega = \begin{cases} -\pi/2 & for \quad x = -\pi \\ x & for \quad -\pi < x < \pi \\ \pi/2 & for \quad x = \pi \\ 0 & for \quad |x| > \pi \end{cases}$$

3) Section 14.1 Problems 7.

$$f(x) = \begin{cases} \sin(x) & for \quad -3\pi \le x \le \pi \\ 0 & for \quad x < -3\pi \quad and \quad x > \pi \end{cases}$$

$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(\xi)\cos(\omega\xi)d\xi = \frac{2\sin(\omega\pi)\sin(2\omega\pi)}{\pi(\omega^2-1)}$$

$$B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\omega\xi)d\xi = \frac{1}{\pi}\int_{-3\pi}^{\pi} \sin(\xi)\sin(\omega\xi)d\xi = \frac{2\cos(\omega\pi)\sin(2\omega\pi)}{\pi(\omega^2-1)}$$

➔The Fourier integral of f is

$$\int_0^\infty \frac{2}{\pi(\omega^2-1)}[\sin(\omega\pi)\sin(2\omega\pi)\cos(\omega x) - \cos(\omega\pi)\sin(2\omega\pi)\sin(\omega x)]d\omega = f(x) \quad for \ all \ x$$

4) Section 14.2 Problems 1.

$$f(x) = \begin{cases} x^2 & for \quad 0 \le x \le 10 \\ 0 & for \quad x > 10 \end{cases}$$

By Definition 14.1 Fourier Cosine Integral

The Fourier cosine integral of f is

$$\int_0^\infty A_\omega \cos(\omega x)d\omega$$

with

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\cos(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\int_0^{10} \xi^2\cos(\omega\xi)d\xi = \frac{2}{\pi}\left[\frac{2\xi}{\omega^2}\cos(\omega\xi) + \left(\frac{\xi^2}{\omega} - \frac{2}{\omega^3}\right)\sin(\omega\xi)\right]\Bigg|_0^{10}$$

$$= \frac{2}{\pi}\left[\frac{20}{\omega^2}\cos(10\omega) + \left(\frac{100}{\omega} - \frac{2}{\omega^3}\right)\sin(10\omega)\right]$$

$$= \frac{4}{\pi\omega^3}\left[10\omega\cos(10\omega) + \left(50\omega^2 - 1\right)\sin(10\omega)\right]$$

The Fourier cosine integral of f converges to x^2 for $0 \le x < 10$, to 50 at $x = 10$, to 0 for $x > 10$

By Definition 14.2 Fourier Sine Integral

The Fourier sine integral of f is

$$\int_0^\infty B_\omega \sin(\omega x)d\omega$$

with

$$B_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\int_0^{10} \xi^2\sin(\omega\xi)d\xi = \frac{2}{\pi}\left[\frac{2\xi}{\omega^2}\sin(\omega\xi) - \left(\frac{\xi^2}{\omega} - \frac{2}{\omega^3}\right)\cos(\omega\xi)\right]\Bigg|_0^{10}$$

$$= \frac{2}{\pi}\left[\frac{20}{\omega^2}\sin(10\omega) - \left(\frac{100}{\omega} - \frac{2}{\omega^3}\right)\cos(10\omega) - \frac{2}{\omega^3}\right]$$

$$= \frac{4}{\pi\omega^3}\left[10\omega\sin(10\omega) - \left(50\omega^2 - 1\right)\cos(10\omega) - 1\right]$$

The Fourier sine integral of f converges to x^2 for $0 \le x < 10$, to 50 at $x = 10$, to 0 for $x > 10$

5) Section 14.2 Problems 9.

$f(x) = \begin{cases} k & for \quad 0 \le x \le c \\ 0 & for \quad x > c \end{cases}$ in which k is constant and c is positive constant.

By Definition 14.1 Fourier Cosine Integral

The Fourier cosine integral of f is

$$\int_0^\infty A_\omega \cos(\omega x)d\omega$$

with

$$A_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\cos(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\int_0^c k\cos(\omega\xi)d\xi = \frac{2}{\pi}\left[\frac{k}{\omega}\sin(\omega\xi)\right]\Big|_0^c$$

$$= \frac{2k}{\pi\omega}\sin(c\omega)$$

The Fourier cosine integral of f converges to k for $0 \le x < c$, to $k/2$ at $x = c$, to 0 for $x > c$

By Definition 14.2 Fourier Sine Integral

The Fourier cosine integral of f is

$$\int_0^\infty B_\omega \cos(\omega x)d\omega$$

with

$$B_\omega = \frac{2}{\pi}\int_0^\infty f(\xi)\sin(\omega\xi)d\xi$$

$$= \frac{2}{\pi}\int_0^c k\sin(\omega\xi)d\xi = \frac{2}{\pi}\left[-\frac{k}{\omega}\cos(\omega\xi)\right]\Big|_0^c$$

$$= \frac{2k}{\pi\omega}[1 - \cos(c\omega)]$$

The Fourier sine integral of f converges to to 0 at $x = 0$, to k for $0 < x < c$, to $k/2$ at $x = c$, to 0 for $x > c$

6) Prove Theorem 14.3

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt$$

(1) If $a > 0$

Set $at = s$, $\Longrightarrow adt = ds$

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt = \int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{a} = \int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{|a|}$$

(2) If $a < 0$ $(a = -|a|)$

Set $at = s, \Longrightarrow -|a|t = s$, $\Longrightarrow -|a|dt = ds$

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt = -\int_{\infty}^{-\infty} f(s)e^{-iws/a}\frac{ds}{|a|} = \int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{|a|}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(at)e^{-iwt}dt = \frac{1}{|a|}\int_{-\infty}^{\infty} f(s)e^{-i\frac{w}{a}s}ds = \frac{1}{|a|}\hat{f}\left(\frac{w}{a}\right)$$

7) Prove Theorem 14.5

$\hat{f}(w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dt$ ➔ $\hat{f}(t) = \int_{-\infty}^{\infty} f(s)e^{-its}\,ds$

$F\left[\hat{f}(t)\right] = \int_{-\infty}^{\infty} \hat{f}(t)e^{-iwt}\,dt = 2\pi f(-w)$

Note that we have the Fourier integral of f representing f as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwt}\,dw$$