Engineering Mathematics II Hw#7 日河工 2B

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1) Section 8.3 Problems 1.

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

$$P_A(\lambda) = \begin{vmatrix} \lambda - 4 & 2 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda - 1) - 4 = \lambda^2 - 5\lambda$$

(1)Eigenvalues of A ➔ $P_A(\lambda) = \lambda^2 - 5\lambda = 0 ==> \lambda = 0, \quad \lambda = 5$

(2)Find **an** eigenvector associated with $\lambda = 0$

➔ $(0 \cdot I_2 - A)X = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = O$ ➔ $\begin{matrix} -4x_1 + 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{matrix}$

➔ $x_1 = \alpha, \ x_2 = 2\alpha$ (Note only one independent equation for two variables)

➔ $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Find **an** eigenvector associated with $\lambda = 5$

➔ $(5 \cdot I_2 - A)X = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = O$ ➔ $\begin{matrix} x_1 + 2x_2 = 0 \\ 2x_1 + 4x_2 = 0 \end{matrix}$

➔ $x_1 = \beta, \ x_2 = -\beta / 2$ (Note only one independent equation for two variables)

➔ $V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(3) $V_1 \bullet V_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} = (1) \cdot (-2) + (2) \cdot (1) = 0$ ➔ $V_1, \quad V_2$ are orthogonal.

(4) $V_1, \quad V_2$ are orthogonal➔linearly independent eigenvectors

$Q = [V_1 \quad V_2] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ ➔ $Q^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

➔ $Q^{-1}AQ = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 0 & 0 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$

Note that, if $P = \begin{bmatrix} \frac{V_1}{\|V_1\|} & \frac{V_2}{\|V_2\|} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$, an orthogonal matrix

➔ $P^{-1} = P^t = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$$\Rightarrow P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & 0 \\ -10 & 5 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

2) Section 8.3 Problems 5.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$P_A(\lambda) = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda+2 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = \lambda(\lambda+2)(\lambda-3) - (\lambda-3) = (\lambda-3)(\lambda^2+2\lambda-1)$$

(1)Eigenvalues of A $\Rightarrow P_A(\lambda) = 0 \Longrightarrow \lambda_1 = 3, \quad \lambda_2 = -1+\sqrt{2}, \quad \lambda_3 = -1-\sqrt{2}$

(2)Find **an** eigenvector associated with $\lambda_1 = 3$

$$\Rightarrow (3\cdot I_3 - A)X = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O \;\Rightarrow\; \begin{matrix} 3x_1 - x_2 = 0 \\ -x_1 + 5x_2 = 0 \end{matrix} \;\Rightarrow\; X = \begin{Bmatrix} 0 \\ 0 \\ \alpha \end{Bmatrix}$$

Note TWO linearly independent equations for THREE variables

$$\Rightarrow V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_2 = -1+\sqrt{2}$

$$\Rightarrow (\lambda_2 \cdot I_3 - A)X = \begin{pmatrix} -1+\sqrt{2} & -1 & 0 \\ -1 & 1+\sqrt{2} & 0 \\ 0 & 0 & -4+\sqrt{2} \end{pmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O$$

$$\Rightarrow \begin{matrix} (-1+\sqrt{2})x_1 - x_2 = 0 \\ -x_1 + (1+\sqrt{2})x_2 = 0 \\ (-4+\sqrt{2})x_3 = 0 \end{matrix} \;\Rightarrow\; X = \begin{Bmatrix} (1+\sqrt{2})\beta \\ \beta \\ 0 \end{Bmatrix}$$

$$\Rightarrow V_2 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_3 = -1-\sqrt{2}$

$$\Rightarrow (\lambda_3 \cdot I_3 - A)X = \begin{pmatrix} -1-\sqrt{2} & -1 & 0 \\ -1 & 1-\sqrt{2} & 0 \\ 0 & 0 & -4-\sqrt{2} \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = O$$

$$\begin{aligned} &(-1-\sqrt{2})x_1 - x_2 = 0 \\ \Rightarrow &-x_1 + (1-\sqrt{2})x_2 = 0 \\ &(-4-\sqrt{2})x_3 = 0 \end{aligned} \Rightarrow X = \begin{Bmatrix} (1-\sqrt{2})\beta \\ \beta \\ 0 \end{Bmatrix}$$

$$\Rightarrow V_3 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix}$$

(3) $V_1 \bullet V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^t \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix} = 0$, $V_1 \bullet V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^t \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix} = 0$,

$V_2 \bullet V_3 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} 1-\sqrt{2} \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow V_1, \quad V_2, \quad V_3$ are orthogonal.

(4) $V_1, \quad V_2, \quad V_3$ are orthogonal $\Rightarrow$ linearly independent eigenvectors

$$P = \begin{bmatrix} \frac{V_1}{\|V_1\|} & \frac{V_2}{\|V_2\|} & \frac{V_3}{\|V_3\|} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ 0 & \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ 1 & 0 & 0 \end{bmatrix}$$, an orthogonal matrix

$$\Rightarrow P^{-1} = P^t$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1+\sqrt{2} & 0 \\ 0 & 0 & -1-\sqrt{2} \end{bmatrix}$$

3) Section 8.3 Problems 11.

$$A=\begin{pmatrix}0&0&0&0\\0&1&-2&0\\0&-2&1&0\\0&0&0&0\end{pmatrix}$$

$$P_A(\lambda)=\begin{vmatrix}\lambda&0&0&0\\0&\lambda-1&2&0\\0&2&\lambda-1&0\\0&0&0&\lambda\end{vmatrix}=\lambda(\lambda-1)(\lambda-31)\lambda$$

(1)Eigenvalues of A ➔ $P_A(\lambda)=0 ==> \lambda_1=0,\quad \lambda_2=0,\quad \lambda_3=-1,\quad \lambda_4=3$

(2)Find **an** eigenvector associated with $\lambda_1=0$, $\lambda_2=0$

$$\Rightarrow (0\cdot I_4-A)X=\begin{pmatrix}0&0&0&0\\0&-1&2&0\\0&2&-1&0\\0&0&0&0\end{pmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=O \Rightarrow \begin{matrix}-x_2+2x_3=0\\2x_2-x_3=0\end{matrix}$$

$$\Rightarrow X=\begin{Bmatrix}\alpha\\0\\0\\\beta\end{Bmatrix}=\alpha\begin{Bmatrix}1\\0\\0\\0\end{Bmatrix}+\beta\begin{Bmatrix}0\\0\\0\\1\end{Bmatrix}$$

$$\Rightarrow V_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad V_2=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_3=-1$

$$\Rightarrow (-1\cdot I_4-A)X=\begin{pmatrix}-1&0&0&0\\0&-2&2&0\\0&2&-2&0\\0&0&0&-1\end{pmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}=O \Rightarrow \begin{matrix}-x_1=0\\-2x_2+2x_3=0\\2x_2-2x_3=0\\-x_4=0\end{matrix}$$

$$\Rightarrow X=\begin{Bmatrix}0\\\alpha\\\alpha\\0\end{Bmatrix}=\alpha\begin{Bmatrix}0\\1\\1\\0\end{Bmatrix}\quad \Rightarrow V_3=\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

Find **an** eigenvector associated with $\lambda_4 = 3$

$$\Rightarrow (3 \cdot I_4 - A)X = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = O \Rightarrow \begin{matrix} 3x_1 = 0 \\ 2x_2 + 2x_3 = 0 \\ 2x_2 + 2x_3 = 0 \\ 3x_4 = 0 \end{matrix}$$

$$\Rightarrow X = \begin{Bmatrix} 0 \\ \alpha \\ -\alpha \\ 0 \end{Bmatrix} = \alpha \begin{Bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{Bmatrix} \quad \Rightarrow V_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

(3) $V_1 \bullet V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$, $V_1 \bullet V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$, $V_1 \bullet V_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$

$V_2 \bullet V_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0$, $V_2 \bullet V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$, $V_3 \bullet V_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} = 0$

$\Rightarrow V_1, \; V_2, \; V_3, \; V_4$ are orthogonal.

(4) $V_1, \; V_2, \; V_3, \; V_4$ are orthogonal $\Rightarrow$ linearly independent eigenvectors

$$P = \begin{bmatrix} \frac{V_1}{\|V_1\|} & \frac{V_2}{\|V_2\|} & \frac{V_3}{\|V_3\|} & \frac{V_4}{\|V_4\|} \end{bmatrix}$$, an orthogonal matrix $\Rightarrow P^{-1} = P^t$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$