國立台灣海洋大學九十三學年度 第二學期 工程數學(二) 第二次期中考解答

1.

(1) $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt=\int_{-\infty}^{\infty} H(t)e^{-at}e^{-i\omega t}dt=\int_{0}^{\infty} e^{-(i\omega+a)t}dt=\frac{-1}{i\omega+a}e^{-(i\omega+a)t}\Big|_0^{\infty}=\frac{1}{a+i\omega}$

(2) $\mathcal{F}[e^{-a|t|}]=\mathcal{F}[H(t)e^{-at}+H(-t)e^{at}]=\mathcal{F}[H(t)e^{-at}]+\mathcal{F}[H(-t)e^{at}]=\mathrm{F}(\omega)+\mathrm{F}(-\omega)=\frac{1}{a+i\omega}+\frac{1}{a-i\omega}=\frac{2a}{a^2+\omega^2}$

($\because$ time reversal：$\mathcal{F}[f(-t)]=\mathrm{F}(-\omega)$)

(3) Modulation：$\mathcal{F}[f(t)\sin at]=\frac{i}{2}[F(\omega+a)-F(\omega-a)]$

Time shifting：$\mathcal{F}[f(t-t_0)]=e^{-i\omega t_0}$ $\quad\therefore y(t)=e^{-8}H(t-4)e^{-2(t-4)}\sin(t-4)$

其中 $\mathcal{F}[H(t)e^{-2t}]=\frac{1}{2ti\omega}$, $\quad\mathcal{F}[H(t)e^{-2t}\sin t]=\frac{i}{2}[\frac{1}{2+i(\omega+1)}-\frac{1}{2+i(\omega-1)}]$,

$\mathcal{F}[H(t-4)e^{-2(t-4)}\sin(t-4)]=e^{-4i\omega}\frac{i}{2}[\frac{1}{2+i(\omega+1)}-\frac{1}{2+i(\omega-1)}]$

$\therefore G(\omega)=\mathcal{F}[e^{-8}H(t-4)e^{-2(t-4)}\sin(t-4)]=e^{-8}e^{-4i\omega}\frac{i}{2}[\frac{1}{2+i(\omega+1)}-\frac{1}{2+i(\omega-1)}]$

(4) $\delta(t)=\lim_{a\to 0}\frac{1}{2a}[H(t+a)-H(t-a)]$

$\mathcal{F}[\delta(t)]=\lim_{a\to 0}\{\frac{1}{2a}\mathcal{F}[H(t+a)-H(t-a)]\}=\lim_{a\to 0}\{\frac{1}{2a}\int_{-a}^{a}e^{-i\omega t}dt\}=\lim_{a\to 0}\{\frac{1}{2a}[\frac{-1}{i\omega}e^{-i\omega t}\Big|_{-a}^{a}]\}$

$=\lim_{a\to 0}\{\frac{1}{2ai\omega}[e^{ia\omega}-e^{-ia\omega}]\}=\lim_{a\to 0}[\frac{1}{2a}\frac{2\sin(a\omega)}{\omega}]=\lim_{a\to 0}\frac{\sin(a\omega)}{a\omega}=\lim_{a\to 0}\frac{\omega\cos(a\omega)}{\omega}=1$

(5) $y''+3y'+2y=\delta(t-5)$, $\quad(-\omega^2+3i\omega+2)Y(\omega)=e^{-5i\omega}$, $\quad Y(\omega)=\frac{e^{-5i\omega}}{(1+i\omega)(2+i\omega)}=\frac{e^{-5i\omega}}{1+i\omega}-\frac{e^{-5i\omega}}{2+i\omega}$,

$\therefore y_p(t)=H(t-5)e^{-(t-5)}-H(t-5)e^{-2(t-5)}$

(6) $y''+3y'+2y=1+te^{-4t}$, $\quad(-\omega^2+3i\omega+2)Y(\omega)=\frac{1}{4+i\omega}$,

$Y(\omega)=\frac{1}{(4+i\omega)(1+i\omega)(2+i\omega)}=\frac{1}{6(4+i\omega)}+\frac{1}{3(1+i\omega)}-\frac{1}{2(2+i\omega)}$

$$\therefore y_p(t) = \frac{1}{6}H(t)e^{-4t} + \frac{1}{3}H(t)e^{-t} - \frac{1}{2}H(t)e^{-2t}$$

(7) $y'' + 3y' + 2y = H(t)e^{-4t} * \delta(t)$, $\quad (-\omega^2 + 3i\omega + 2)Y(\omega) = \mathcal{F}[H(t)e^{-4t}]\mathcal{F}[\delta(t)]$,

$$Y(\omega) = \frac{1}{(4+i\omega)(1+i\omega)(2+i\omega)}, \quad \therefore y_p(t) = \frac{1}{6}H(t)e^{-4t} + \frac{1}{3}H(t)e^{-t} - \frac{1}{2}H(t)e^{-2t}$$

2.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} \cos(\omega t) f(t)dt - i\int_{-\infty}^{\infty} \sin(\omega t) f(t)dt = R(\omega) + iX(\omega)$$

其中 $R(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) f(t)dt$, $\quad X(\omega) = -\int_{-\infty}^{\infty} \sin(\omega t) f(t)dt$

而 $A(\omega) = \frac{1}{\pi}\int f(t)\cos(\omega t)dt$, $\quad B(\omega)\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$

$\therefore R(\omega) = \pi A(\omega)$, $\quad X(\omega) = -\pi B(\omega)$

3.

a) omitted

b) yes

c) $\int_{-\infty}^{\infty} |f(t)|dt = \int_{-c}^{c} k\ dt = 2kc$ ➔ exists and is finite

d) $A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\omega\xi)d\xi = \frac{1}{\pi}\int_{-c}^{c} k\cos(\omega\xi)d\xi = \frac{k}{\pi}\frac{\sin(\omega\xi)}{\omega}\Bigg|_{-c}^{c} = \frac{2k\sin(\omega c)}{\pi\omega}$

$B_\omega = 0$

e) $\int_{0}^{\infty} [A_\omega\cos(\omega t) + B_\omega\sin(\omega t)]\ d\omega = \frac{2k}{\pi}\int_{0}^{\infty} \frac{\sin(\omega c)}{\omega}\cos(\omega t)d\omega$

By applying the convergence theorem,

$$\int_{0}^{\infty} [A_\omega\cos(\omega t) + B_\omega\sin(\omega t)]\ d\omega = \frac{2k}{\pi}\int_{0}^{\infty} \frac{\sin(\omega c)}{\omega}\cos(\omega t)d\omega = \begin{cases} k & for \quad -c < t < c \\ k/2 & for \quad t = \pm c \\ 0 & for \quad |t| > c \end{cases}$$

f) if $k = \frac{1}{2c}$ and $c \to 0$ ➔ $f(t)$ is the Dirac Delta function

➔ $C_\omega = 1$ ➔ the complex Fourier integral of $f(t)$ is $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega = \delta(t)$

Note: get the “5 scores” for free !

4.

a) $f(x) = xe^{-|x|}$

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} xe^{-|x|}e^{-i\omega x}dx = \int_{-\infty}^{0} xe^{x}e^{-i\omega x}dx + \int_{0}^{\infty} xe^{-x}e^{-i\omega x}dx$$

$$= x\frac{e^{(1-i\omega)x}}{1-i\omega}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{(1-i\omega)x}}{1-i\omega}dx + x\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-(1+i\omega)x}}{-(1+i\omega)}dx$$

$$= -\frac{e^{(1-i\omega)x}}{(1-i\omega)^2}\Big|_{-\infty}^{0} - \frac{e^{-(1+i\omega)x}}{(1+i\omega)^2}\Big|_{0}^{\infty}$$

$$= -\frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} = \frac{-4i\omega}{(1+\omega^2)^2}$$

b)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} |x|e^{-|x|}e^{-i\omega x}dx = \int_{-\infty}^{0} -xe^{x}e^{-i\omega x}dx + \int_{0}^{\infty} xe^{-x}e^{-i\omega x}dx$$

$$= -x\frac{e^{(1-i\omega)x}}{1-i\omega}\Big|_{-\infty}^{0} + \int_{-\infty}^{0} \frac{e^{(1-i\omega)x}}{1-i\omega}dx + x\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-(1+i\omega)x}}{-(1+i\omega)}dx$$

$$= \frac{e^{(1-i\omega)x}}{(1-i\omega)^2}\Big|_{-\infty}^{0} - \frac{e^{-(1+i\omega)x}}{(1+i\omega)^2}\Big|_{0}^{\infty}$$

$$= \frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} = \frac{2(1-\omega^2)}{(1+\omega^2)^2}$$