海洋大學河海工程學系九十四學年度 第二學期 ■期中 □小考 考試命題紙

考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學二	二 A, B	4 月 14 日	111	■需 □不需	陳桂鴻 呂學育	第一次大考

1. $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$; Evaluate $\oint_C \vec{F}\cdot dr$

(a)C is shown as Fig1(a). (Hint: Using direct integral) (7%)

(b)C is shown as Fig1(b). (Hint: Using Green's theorem) (7%)

(c) C is shown as Fig1(c). (Hint: Using Green's theorem) (6%)

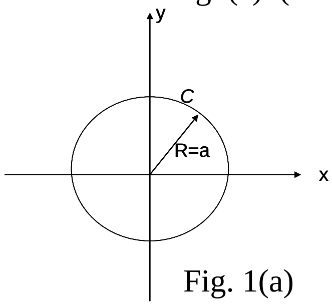

Fig. 1(a)

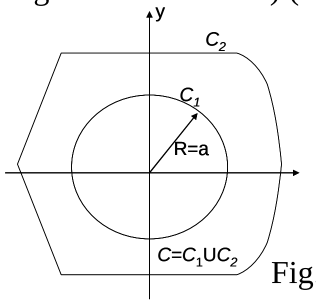

Fig. 1(b)

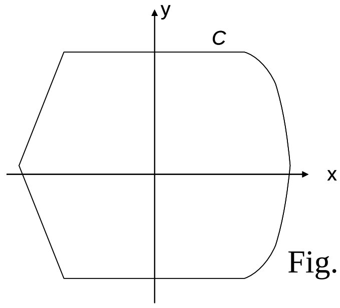

Fig. 1(c)

Ans: (a) $x = a\cos\theta$, $dx = -a\sin\theta d\theta$, $y = a\sin\theta$, $dy = a\cos\theta d\theta$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \frac{-a\sin\theta}{a^2}(-a\sin\theta d\theta) + \frac{a\cos\theta}{a^2}(-a\cos\theta d\theta)$$

$$= \int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = 2\pi$$

(b) $P = \frac{-y}{x^2+y^2}$, $\frac{\partial P}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}$, $Q = \frac{x}{x^2+y^2}$, $\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = 0$$

(c) $\oint_C \vec{F}\cdot d\vec{r} = \int_{C'} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 2\pi$

2. $\vec{F} = y^3\vec{i} - x^3\vec{j} + z^3\vec{k}$; C is the trace of the cylinder $x^2+y^2=1$ in the plane $x+y+z=1$.

(a) Show that the force is conservative or nonconservative. (5%)

(b) Use Stokes's theorem to evaluate $\oint_C \vec{F}\cdot dr$. (15%)

Ans: (a) $\nabla\times\vec{F} = (-3x^2-3y^2)\vec{k}$, $\vec{F}$ is nonconservative.

(b) $\phi = x+y+z-1=0$, $\nabla_\phi = \vec{i}+\vec{j}+\vec{z}$, $\vec{n} = \frac{\nabla_\phi}{|\nabla_\phi|} = \frac{1}{\sqrt{3}}(\vec{i}+\vec{j}+\vec{z})$, $ds = \sqrt{3}dA$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}ds = \iint_R -3(x^2+y^2)dA = -3\int_0^{2\pi}\int_0^1 r^2 r dr d\theta = \frac{-3}{2}\pi$$

3. The given vector field $\vec{F}(x,y,z) = (x\vec{i}+y\vec{j}+z\vec{k})/(x^2+y^2+z^2)$, S is the region bounded by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(a). Find $\nabla\cdot\vec{F}, \nabla\times\vec{F}$. (5%)

(b). Find the normal vector $\vec{n}$ of S. (5%)

(c). Use the divergence theorem to find the outward flux $\iint_S (\vec{F}\cdot\vec{n})dS$ of $\vec{F}$. (10%)

Ans: (a) $\nabla\cdot\vec{F}=\frac{1}{x^2+y^2+z^2}$, $\nabla\times\vec{F}=0$

(b) $\phi=x^2/a^2+y^2/b^2+z^2/c^2-1$, $\nabla_\phi=\frac{2x}{a^2}\vec{i}+\frac{2y}{b^2}\vec{j}+\frac{2z}{c^2}\vec{k}$, $|\nabla_\phi|=2\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$

$$\vec{n}=\frac{\nabla_\phi}{|\nabla_\phi|}=(\frac{x}{a^2}\vec{i}+\frac{y}{b^2}\vec{j}+\frac{z}{c^2}\vec{k})/(\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}})$$

(c) $\iint_S \vec{F}\cdot\vec{n}dS=\iiint_D div\vec{F}dV=\int_0^{2\pi}\int_0^{\pi}\int_a^b\frac{1}{\rho^2}\rho^2\sin\phi d\rho d\phi d\theta=4\pi(b-a)$

4. Suppose $\vec{r}(t)=t^2\vec{i}+(t^3-2t)\vec{j}+(t^2-5t)\vec{k}$ is the position vector of a moving particle. What are its speed, velocity, acceleration, curvature and tangent line at the point (0,0,0) ? (15 scores)

Ans: $\vec{r}(t)=t^2\vec{i}+(t^3-2t)\vec{j}+(t^2-5t)\vec{k}$

$\vec{v}(t)=2t\vec{i}+(3t^2-2)\vec{j}+(2t-5)\vec{k}$

$\vec{a}(t)=2\vec{i}+6t\vec{j}+2\vec{k}$

Speed: $|\vec{v}(0)|=\sqrt{(-2)^2+(-5)^2}=\sqrt{29}$

velocity: $\vec{v}(0)=-2\vec{j}-5\vec{k}$

acceleration: $\vec{a}(0)=2\vec{i}+2\vec{k}$

curvature: $\kappa=\frac{|\vec{v}(0)\times\vec{a}(0)|}{|\vec{v}(0)|^3}=\frac{|-4\vec{i}-10\vec{j}+4\vec{k}|}{\sqrt{29}^3}=\frac{2}{29}\sqrt{\frac{33}{29}}$

tangent line through $(0,0,0)$: $x=0+0t,\ y=0+(-2)t,\ z=0+(-5)t$

5. If S is the portion of the plane $x+2y+3z=6$ in the first octant. For $\vec{F}=y\vec{i}+z\vec{j}+x\vec{k}$

(1) find the area of S (5 scores)

(2) find the upper unit normal of S (5 scores)

(3) use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$,where the curve C is the boundary of S and C is oriented counterclockwise as viewed from above. (10 scores)

Ans: (1) $z=f(x,y)=\frac{6-x-2y}{3}$

$f_x(x,y)=-\frac{1}{3}$, $f_y(x,y)=-\frac{2}{3}$, $dS=\sqrt{1+f_x^2+f_y^2}=\sqrt{\frac{14}{9}}dA$

$$A=\int_0^6\int_0^{\frac{6-x}{2}}\sqrt{\frac{14}{9}}dydx=\sqrt{\frac{14}{9}}\int_0^6\frac{6-x}{2}dx=\frac{1}{2}\sqrt{\frac{14}{9}}(6x-\frac{1}{2}x^2)\Big|_0^6=3\sqrt{14}$$

(2) $g(x,y,z)=x+2y+3z=6$, $\vec{n}=\dfrac{\nabla_g}{|\nabla_g|}=\dfrac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}}$

(3) $\oint_C \vec{F}\cdot d\vec{r}=\iint_S(\nabla\times\vec{F})\cdot\vec{n}dS=\iint_S(-\vec{i}-\vec{j}-\vec{k})(\dfrac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}})dS=-\dfrac{6}{\sqrt{14}}\iint_R\dfrac{\sqrt{14}}{3}dA=-18$

6. If S is the surface of the region bounded by $x^2+y^2=4$, $z=\sqrt{16-x^2-y^2}$, $z=0$.

$\vec{F}=-y^3\vec{i}-x^3\vec{j}+z^3\vec{k}$

(1) find the volume of the solid bounded by $x^2+y^2=4$, $z=\sqrt{16-x^2-y^2}$, $z=0$. (10 scores)

(2) use the divergence theorem to find the outward flux $\iint_S\left(\vec{F}\cdot\vec{n}\right)dS$ (15 scores)

Ans: (1) $V=\iiint_D dxdydz=\int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}}rdzdrd\theta=\int_0^{2\pi}\int_0^2 r\sqrt{16-r^2}drd\theta$

$=\int_0^{2\pi}-\dfrac{1}{3}(12\sqrt{12}-64)d\theta=-16\sqrt{3}\pi+\dfrac{128}{3}\pi$

(2) $div\,\vec{F}=\dfrac{\partial(-y^3)}{\partial x}+\dfrac{\partial(-x^3)}{\partial y}+\dfrac{\partial(z^3)}{\partial z}=3z^2$

Using cylindrical coordinates, $\iint_S\vec{F}\cdot\vec{n}dS=\iiint_D div\,\vec{F}dV=\int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}}3z^2rdzdrd\theta$

$=\int_0^{2\pi}\int_0^2 rz^3\Big|_0^{\sqrt{16-r^2}}drd\theta=\int_0^{2\pi}\int_0^2 r(16-r^2)^{3/2}drd\theta$

$=\int_0^{2\pi}-\dfrac{1}{5}(16-r^2)^{5/2}\Big|_0^2 d\theta=\int_0^{2\pi}-\dfrac{1}{5}(12^{5/2}-4^5)d\theta=\int_0^{2\pi}\dfrac{1}{5}(1024-144\sqrt{12})d\theta=\dfrac{2\pi}{5}(1024-288\sqrt{3})$