海洋大學河海工程學系九十四學年度 第二學期 ■期中 □小考 考試命題紙

考 試 科 目	開課系級	考試日期	印製份數	答案紙	命題教師	備 註
工程數學二	二 A, B	6 月 2 日	110	■ 需 □不需	陳桂鴻 呂學育	第三次大考

1. (1) Proof $F[\ f(t)*g(t)\ ]=F(\omega)G(\omega)$, where $*$ is convolution operator.(10%)

(2) $f(t)=e^{-a|t|}, a>0$, compute $F[\ f(t)\ ]$. (3%)

(3) $g(t)=\delta(t-1)$, compute $F[\ g(t)\ ]$. (4%)

(4) Compute $F[\ f(t)*g(t)\ ]$=? (3%)

(5) Solve y_p of $y''(t)+2y(t)=f(t)*g(t)$. $\left(F^{-1}\left[\dfrac{1}{2-w^2}\right]=\dfrac{\pi\sin\sqrt{2}t}{\sqrt{2}}\text{ by using Residue theorem}\right)$ (10%)

ANS (1) $F[\ f(t)*g(t)\ ]=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau]e^{-i\omega t}dt=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(t-\tau)e^{-i\omega(t-\tau)}g(\tau)e^{-i\omega\tau}d\tau]dt$

$$=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}\overline{f}(t-\tau)\overline{g}(\tau)d\tau]dt=\int_{-\infty}^{\infty}\overline{f}*\overline{g}dt=\int_{-\infty}^{\infty}\overline{f(t)}dt\int_{-\alpha}^{\infty}\overline{g(t)}dt=\int_{-\infty}^{\infty}\overline{f(t)}e^{-i\omega t}dt\int_{-\alpha}^{\infty}\overline{g(t)}e^{-i\omega t}dt$$

$$=F(\omega)G(\omega)$$

(2) $F[\ f(t)\ ]=\int_{-\infty}^{\infty}e^{-a|t|}e^{-i\omega t}dt=\dfrac{2a}{a^2+\omega^2}$

(3) $F[\ g'(t)\ ]=\int_{-\infty}^{\infty}\delta(t)e^{-i\omega t}dt=1$, $\quad F[\ g(t)\ ]=F[\ g'(t-1)\ ]=e^{-i\omega}$

(4) $F[\ f(t)*g(t)\ ]=F(\omega)G(\omega)=\dfrac{2a}{a^2+\omega^2}e^{-i\omega}$

(5) $(-\omega^2+2)Y(\omega)=\dfrac{2a}{a^2+\omega^2}e^{-i\omega}$, $\quad Y(\omega)=\dfrac{2a}{a^2+\omega^2}\dfrac{1}{(-\omega^2+2)}e^{-i\omega}$, $\quad Y(\omega)=\left[\dfrac{\frac{2a}{a^2+2}}{a^2+\omega^2}+\dfrac{\frac{2a}{a^2+2}}{(-\omega^2+2)}\right]e^{-i\omega}$

$$y=F^{-1}\left[\frac{\frac{2a}{a^2+2}}{a^2+\omega^2}e^{-i\omega}\right]+F^{-1}\left[\frac{\frac{2a}{a^2+2}}{(-\omega^2+2)}e^{-i\omega}\right]=\frac{1}{a^2+2}e^{-a|t-1|}+\frac{2a}{a^2+2}\frac{\pi\sin\sqrt{2}(t-1)}{\sqrt{2}}$$

2. (1) Derive the Fourier transform formulation by using the complex Fourier series. (10%)

(2) Write the formulations of transform pairs according to your knowledge. (5%)

(3) Proof the relations of $F(\omega)$ and $A(\omega), B(\omega)$, where $F(\omega)$ is the Fourier transform of $f(t)$, and $A(\omega), B(\omega)$ are the Fourier integral coefficients of $f(t)$. (15%)

ANS (1) $f(x)=\sum_{n=-\infty}^{\infty}C_n e^{i\omega_n x}$ $\quad$ where $C_n=\dfrac{1}{T}\int_{-p}^{p}f(x)e^{-i\omega_n x}dx$

$$f(x)=\sum_{n=-\infty}^{\infty}[\frac{1}{T}\int_{-p}^{p}f(x)e^{-i\omega_n x}dx]e^{i\omega_n x}=\sum_{n=-\infty}^{\infty}[\frac{1}{T}\int_{-p}^{p}f(x)e^{-i\omega_n x}dx]e^{i\omega_n x}\frac{1}{\Delta\omega}\Delta\omega$$

When $T\to\infty, x\in(-\infty,\infty)$; $\omega_n=\dfrac{2n\pi}{T}\to\omega$; $\Delta\omega=\dfrac{2\pi}{T}\to d\omega$; $\sum_{n=-\infty}^{\infty}\to\int_{-\infty}^{\infty}$, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx]e^{i\omega x}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}d\omega\text{, where }F(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$$

(2)

Fourier transform	**Inverse Fourier transform**
$F(x)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$	$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}d\omega$
Fourier cosine transform	**Inverse Fourier cosine transform**
$A(\omega)=\int_{0}^{\infty}f(x)\cos\omega x dx$	$f(x)=\frac{2}{\pi}\int_{0}^{\infty}A(\omega)\cos\omega x d\omega$
Fourier sine transform	**Inverse Fourier sine transform**
$B(\omega)=\int_{0}^{\infty}f(x)\sin\omega x dx$	$f(x)=\frac{2}{\pi}\int_{0}^{\infty}B(\omega)\sin\omega x d\omega$

(3) $F(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx=\int_{-\infty}^{\infty}f(x)(\cos\omega x-i\sin\omega x)dx=\int_{-\infty}^{\infty}\overline{A}(\omega)-i\overline{B}(\omega)dx$

where $\overline{A}(\omega)=\int_{-\infty}^{\infty}f(x)\cos\omega x dx$, $\overline{B}(\omega)=\int_{-\infty}^{\infty}f(x)\sin\omega x dx$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos\omega_n x+b_n\sin\omega_n x],\quad \omega_n=\frac{2n\pi}{T},\quad a_0=\frac{2}{T}\int_{-T/2}^{T/2}f(x)dx,$$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2}f(x)\cos\omega_n dx,\quad b_n=\frac{2}{T}\int_{-T/2}^{T/2}f(x)\sin\omega_n dx$$

$$f(x)=\frac{1}{2}\frac{2}{T}\int_{-T/2}^{T/2}f(t)dt+\sum_{n=1}^{\infty}[\frac{2}{T}\int_{-T/2}^{T/2}f(t)\cos\omega_n t dt\cos\omega_n x+\frac{2}{T}\int_{-T/2}^{T/2}f(t)\sin\omega_n t dt\sin\omega_n x]$$

$$=\frac{1}{\pi}\int_{0}^{\infty}[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

where $A(\omega)=\int_{-\infty}^{\infty}f(x)\cos\omega x dx$, $B(\omega)=\int_{-\infty}^{\infty}f(x)\sin\omega x dx$

$\overline{A}(\omega)=A(\omega)$, $\overline{B}(\omega)=B(\omega)$, 故得證

3. (1) The Heaviside function $H(t)$ is given by $H(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$

Find the Fourier transform of $f(t)=H(t)e^{-at}$ with a a positive constant. (4 scores)

(2) Determine the value of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{3i\omega}}{4+i\omega}e^{i\omega t}d\omega$ (5 scores)

(3) Find the Fourier transform of the Dirac delta function $\delta(t)$, $F\{\delta(t)\}$ (4 scores)

(4) Find the Fourier transform of $F\{\delta(t)\}$, $F\{F\{\delta(t)\}\}$ (5 scores)

(5) Find the Fourier transform of $\cos(5t)$, $F\{\cos(5t)\}$ (5 scores)

ANS (1) $F\{f(t)\}=\int_{-\infty}^{\infty}H(t)e^{-at}e^{-i\omega t}dt=\int_{0}^{\infty}e^{-(a+i\omega)t}dt=-\frac{1}{a+i\omega}e^{-(a+i\omega)t}\Big|_0^{\infty}=\frac{1}{a+i\omega}$

(2) $F^{-1}\left\{\dfrac{1}{4+i\omega}\right\} = f(\mathrm{t}) = \mathrm{H(t)e^{-4t}} \rightarrow \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\dfrac{1}{4+i\omega}e^{i\omega t}d\omega = f(\mathrm{t}) = \mathrm{H(t)e^{-4t}}$

Applying the time time-shifting theorem on $f(t+3)$

$$F\{f(t+3)\} = \frac{e^{3i\omega}}{4+i\omega}$$

Therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{3i\omega}}{4+i\omega}e^{i\omega t}d\omega = f(\mathrm{t}+3) = \mathrm{H(t+3)e^{-4(t+3)}}$$

(3) $F\{\delta(\mathrm{t})\} = \int_{-\infty}^{\infty}\delta(\mathrm{t})e^{-i\omega t}dt = e^{-i\omega t}\big|_{t=0} = 1$

(4) $F\{F\{\delta(t)\}\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

(5) $F\{\cos(5t)\} = \dfrac{1}{2}\left[2\pi\delta(\omega+5)+2\pi\delta(\omega-5)\right] = \pi\delta(\omega+5)+\pi\delta(\omega-5)$

4. (1) Solve $\dfrac{d^2y(t)}{dt^2}+4y(t)=\cos(\omega_0 t)$ using Fourier transform with ω_0 a real number (15 scores)

(2) Discuss your result obtained in (1) if $\omega_0 = 2$ (5 scores)

ANS (1) $F\{\cos(\omega_0 t)\} = \dfrac{1}{2}\left[2\pi\delta(\omega+\omega_0)+2\pi\delta(\omega\text{-}\omega_0)\right] = \pi\delta(\omega+\omega_0)+\pi\delta(\omega\text{-}\omega_0)$

(2) $\dfrac{d^2y(t)}{dt^2}+4y(t)=\cos(\omega_0 t) \rightarrow F\left\{\dfrac{d^2y(t)}{dt^2}+4y(t)\right\} = F\{\cos(\omega_0 t)\}$

$$\rightarrow (\mathrm{i}\omega)^2 F\{y(t)\} + 4F\{y(t)\} = F\{\cos(\omega_0 t)\}$$

$$\rightarrow F\{y(t)\} = \frac{F\{\cos(\omega_0 t)\}}{(\mathrm{i}\omega)^2+4} = \frac{\pi\delta(\omega+\omega_0)+\pi\delta(\omega\text{-}\omega_0)}{4\text{-}\omega^2}$$

$$\rightarrow y(t) = F^{-1}\{F\{y(t)\}\}$$

$$= F^{-1}\left\{\frac{\pi\delta(\omega+\omega_0)+\pi\delta(\omega\text{-}\omega_0)}{4\text{-}\omega^2}\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi\delta(\omega+\omega_0)+\pi\delta(\omega\text{-}\omega_0)}{4\text{-}\omega^2}e^{i\omega t}d\omega$$

$$= \frac{1}{2\pi}\left\{\left[\frac{\pi}{4\text{-}\omega^2}e^{i\omega t}\right]\Big|_{\omega=-\omega_0} + \left[\frac{\pi}{4\text{-}\omega^2}e^{i\omega t}\right]\Big|_{\omega=\omega_0}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\pi}{4\text{-}\omega_0^2}e^{-i\omega_0 t} + \frac{\pi}{4\text{-}\omega_0^2}e^{i\omega_0 t}\right\}$$

$$= \frac{1}{2}\frac{e^{-i\omega_0 t}+e^{i\omega_0 t}}{4\text{-}\omega_0^2} = \frac{\cos\omega_0 t}{4\text{-}\omega_0^2}$$

If $\omega_0 = 2$

$y(t) = \dfrac{\cos\omega_0 t}{4\text{-}\omega_0^2}\Big|_{\omega_0=2} \text{--}> \infty$ ➔Resonance

5. (1) Find the Fourier sine transform of $f(t)=e^{-at}$ with a a positive constant. (5 scores)

(2) If we assume that $f(t)\to 0, f'(t)\to 0$ as $t\to\infty$, solve the boundary-value problem $y''(t)-2y(t)=e^{-t}$, $0<t<\infty$, with $y(0)=y_0$. (12 scores)

ANS (1) $F_s\left\{e^{-at}\right\}=\int_0^{\infty} e^{-at}\sin\omega t\,dt=\frac{\omega}{a^2+\omega^2} \qquad (a>0)$

(2) We take the Fourier sine transform of the ODE

$$\to F_s\left\{y''(t)-2y(t)\right\}=F_s\left\{e^{-t}\right\}\to F_s\left\{y''(t)\right\}-2F_s\left\{y(t)\right\}=F_s\left\{e^{-t}\right\}$$

$$\to -\omega^2 F_s\left\{y(t)\right\}+\omega y_0-2F_s\left\{y(t)\right\}=\frac{\omega}{\omega^2+1}$$

$$\to -F_s\left\{y(t)\right\}\left(\omega^2+2\right)=\frac{\omega}{\omega^2+1}-\omega y_0\to F_s\left\{y(t)\right\}=\frac{\omega y_0}{\left(\omega^2+2\right)}-\frac{\omega}{\left(\omega^2+2\right)\left(\omega^2+1\right)}$$

$$\to y(t)=F^{-1}\left\{\frac{\omega y_0}{\left(\omega^2+2\right)}-\frac{\omega}{\left(\omega^2+2\right)\left(\omega^2+1\right)}\right\}$$

$$\boxed{\frac{1}{\left(\omega^2+2\right)\left(\omega^2+1\right)}=\frac{1}{\left(\omega^2+1\right)}-\frac{1}{\left(\omega^2+2\right)}}$$

$$\to y(t)=F^{-1}\left\{\frac{\omega y_0}{\left(\omega^2+2\right)}+\frac{\omega}{\left(\omega^2+2\right)}-\frac{\omega}{\left(\omega^2+1\right)}\right\}$$

$$=F^{-1}\left\{(y_0+1)\frac{\omega}{\left(\omega^2+2\right)}-\frac{\omega}{\left(\omega^2+1\right)}\right\}$$

$$=(y_0+1)F^{-1}\left\{\frac{\omega}{\left(\omega^2+2\right)}\right\}-F^{-1}\left\{\frac{\omega}{\left(\omega^2+1\right)}\right\}$$

$$=(y_0+1)e^{-\sqrt{2}t}-e^{-t}$$