$$a_2 y'' + a_1 y' + a_0 y = \delta(t)$$

$$L\{a_2 y'' + a_1 y' + a_0 y\} = L\{\delta(t)\} = 1$$
$$a_2[s^2F(s) - sy(0) - y'(0)] + a_1[sF(s) - y(0)] + a_0F(s) = 1$$

$$F(s)[a_2s^2 + a_1s + a_0] + y(0)[-a_2s - a_1] - a_2y'(0) = 1$$
$$F(s)P(s) - Q(s) = 1$$
$$P(s) = a_2s^2 + a_1s + a_0$$
$$Q(s) = y(0)[a_2s + a_1] + a_2y'(0)$$

$$y(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + L^{-1}\left\{\frac{1}{P(s)}\right\} = y_c(t) + y_p(t)$$

If $y(0) = 0,\ y'(0) = 0$ ➔ $Q(s) = 0$

➔ $y(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{P(s)}\right\} = y_\delta(t)$ ➔ **the unit impulse response**

$$a_2 y'' + a_1 y' + a_0 y = f(t)$$

$$L\{a_2 y'' + a_1 y' + a_0 y\} = L\{f(t)\}$$
$$a_2[s^2F(s) - sy(0) - y'(0)] + a_1[sF(s) - y(0)] + a_0F(s) = L\{f(t)\}$$

$$F(s)[a_2s^2 + a_1s + a_0] + y(0)[-a_2s - a_1] - a_2y'(0) = L\{f(t)\}$$
$$F(s)P(s) - Q(s) = L\{f(t)\}$$
$$P(s) = a_2s^2 + a_1s + a_0$$
$$Q(s) = y(0)[a_2s + a_1] + a_2y'(0)$$

$$F(s)P(s) = Q(s) + L\{f(t)\}$$
$$F(s) = \frac{Q(s)}{P(s)} + L\{f(t)\}\frac{1}{P(s)}$$
$$y(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + L^{-1}\left\{L\{f(t)\}\frac{1}{P(s)}\right\}$$

If $y(0) = 0,\ y'(0) = 0$ ➔ $Q(s) = 0$

$$
\begin{aligned}
y(t) &= L^{-1}\{F(s)\} \\
&= L^{-1}\left\{L\{f(t)\}\frac{1}{P(s)}\right\} \\
&= L^{-1}\{L\{f(t) * y_\delta(t)\}\} \\
\Rightarrow \quad &= f(t) * y_\delta(t) \qquad \Rightarrow \text{convolution} \\
&= \int_0^t f(\tau)\ y_\delta(t-\tau)\ d\tau \\
&= y_\delta(t) * f(t) \\
&= \int_0^t y_\delta(\tau)\ f(t-\tau)\ d\tau
\end{aligned}
$$

注意， $y(0)=0,\ \ y^{'}(0)=0$

==

Ex1: $y^{''}+y=\delta(t) \qquad y(0)=0,\ \ y^{'}(0)=0$

$$y(t)=L^{-1}\{F(s)\}=L^{-1}\left\{\frac{Q(s)}{P(s)}\right\}+L^{-1}\left\{\frac{1}{P(s)}\right\}$$

$y(0)=0,\ \ y^{'}(0)=0$ ➔ $Q(s)=0$

$$
\begin{aligned}
y(t) &= L^{-1}\{F(s)\} \\
&= L^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= \sin t
\end{aligned}
$$

➔ $y^{'}(t)=\cos t$ ➔ $y(0)=0,\ \ y^{'}(0)=1$???

Note that, due to the action of the unit impulse on the system with zero initial conditions, there is new initial conditions just after impulse.

$\int_0^\varepsilon y^{''}dt+\int_0^\varepsilon y dt=\int_0^\varepsilon \delta(t)dt$ where ε ➔0 (just after the unit impulse)

➔ $y^{'}(\varepsilon)-y^{'}(0)=\int_0^\varepsilon \delta(t)dt=1$ (注意衝量的作用在於造成速度改變，但位移的變化需要時間，當 ε ➔0 僅瞬間內，所以位移來不及變化)

➔ $y^{'}(\varepsilon)=1$, (ε ➔0 , just after the unit impulse).

==

Ex2: $y'' + y = 0 \qquad y(0)=0,\ y'(0)=1$

Let $y(t)=e^{\lambda t}$ ➔ $y'(t)=\lambda e^{\lambda t}$ ➔ $y''(t)=\lambda^2 e^{\lambda t}$

Substitute into the differential equation➔ $\lambda^2+1=0$ ➔ $\lambda=\pm i$

$y(t)=a\cos t+b\sin t$ ➔ $y(t)=\sin t$ (滿足 $y(0)=0,\ y'(0)=1$)

==

Ex3: $y''+y=\delta(t) \qquad y(0)=0,\ y'(0)=0$

$$y(t)=L^{-1}\{F(s)\}=L^{-1}\left\{\frac{Q(s)}{P(s)}\right\}+L^{-1}\left\{\frac{1}{P(s)}\right\}$$

$y(0)=0,\ y'(0)=0$ ➔ $Q(s)=0$

$$y(t)=L^{-1}\{F(s)\}$$
$$=L^{-1}\left\{\frac{1}{s^2+1}\right\}$$ (滿足新初始條件 $y(0)=0,\ y'(0^+)=1$)
$$=\sin t$$

==

Ex4: $y''+y=A \qquad y(0)=0,\ y'(0)=0$

若直接以待定係數法，可得特解 $y_p=A$

本題補解 $y_c=a\cos t+b\sin t$ ➔ $y=y_c+y_p=a\cos t+b\sin t+A$

➔ $y=-A\cos t+A$ (滿足初始條件 $y(0)=0,\ y'(0)=0$)

$$y(t)=L^{-1}\{F(s)\}=L^{-1}\left\{\frac{Q(s)}{P(s)}\right\}+L^{-1}\left\{\frac{L\{A\}}{P(s)}\right\}$$

$y(0)=0,\ y'(0)=0$ ➔ $Q(s)=0$

$$y(t) = L^{-1}\{F(s)\}$$

$$= L^{-1}\left\{L\{A\}L\left\{\frac{1}{s^2+1}\right\}\right\}$$

$$= L^{-1}\left\{L\left\{A * \frac{1}{s^2+1}\right\}\right\}$$

$$= \int_0^t A\sin(t-\tau)d\tau$$

$$= A\cos(t-\tau)\Big|_0^t = A - A\cos t$$

注意：其中 A 為以待定係數法所求得之特解 y_p ,但 $-A\cos t$ 則為補解 y_c (滿足 $y'' + y = 0$ 及初始條件)

而 $A - A\cos t$ 滿足 $y(0) = 0,\ \ y'(0) = 0$

Ex5.

$$\boxed{a_1 y' + a_0 y = \delta(t-a)}$$

$$L\{a_1 y' + a_0 y\} = L\{\delta(t-a)\} = e^{-as}$$
$$a_1[sF(s) - y(0)] + a_0 F(s) = e^{-as}$$

$$F(s)[a_1 s + a_0] - a_1 y(0) = e^{-as}$$
$$F(s)P(s) - Q(s) = e^{-as}$$
$$P(s) = a_1 s + a_0$$
$$Q(s) = a_1 y(0)$$

$$y(t) = L^{-1}\{F(s)\}$$

$$= L^{-1}\left\{\frac{Q(s)}{P(s)}\right\} + L^{-1}\left\{\frac{e^{-as}}{P(s)}\right\}$$

$$= L^{-1}\left\{\frac{a_1 y(0)}{a_1 s + a_0}\right\} + L^{-1}\left\{\frac{e^{-as}}{a_1 s + a_0}\right\}$$

$$= y(0)e^{-\frac{a_0}{a_1}t}u(t) + \frac{1}{a_1}e^{-\frac{a_0}{a_1}(t-a)}u(t-a)$$