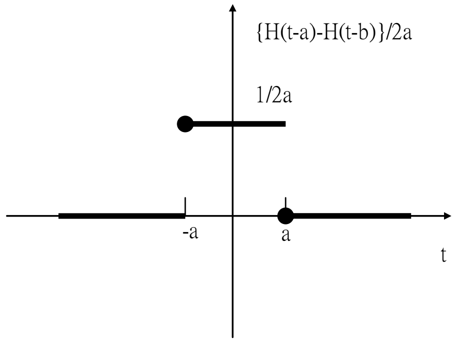
Dirac Delta Function

The **Dirac Delta Function**, also known as the unit impulse function. It is not a true function since it cannot be defined completely by giving the function value for all values of the argument (t).

A Dirac delta function is a pulse of infinite magnitude having infinitely short duration. One way to describe such an object mathematically is to form a short pulse

$$\frac{1}{2a}\left[H(t+a)-H(t-a)\right]$$



Then take the limit as the width of the pulse approaches zero

$$\delta(t)=\lim_{a\to 0}\frac{1}{2a}\left[H(t+a)-H(t-a)\right] \Rightarrow \delta(t)=0 \quad \text{for all} \quad t\neq 0$$

$$\delta(t-c)=\lim_{a\to 0}\frac{1}{2a}\left[H(t-c+a)-H(t-c-a)\right] \Rightarrow \delta(t-c)=0 \quad \text{for all} \quad t\neq c$$

$$\int_{-\infty}^{\infty}\delta(t)dt=\lim_{a\to 0}\frac{1}{2a}\int_{-\infty}^{\infty}\left[H(t+a)-H(t-a)\right]dt$$

$$=\lim_{a\to 0}\frac{1}{2a}\int_{-a}^{a}dt$$

$$=\lim_{a\to 0}\frac{2a}{2a}$$

For any continuous function $f(t)$:

$$\int_{-\infty}^{\infty}\delta(t)f(t)dt=\lim_{a\to 0}\frac{1}{2a}\int_{-\infty}^{\infty}f(t)\left[H(t+a)-H(t-a)\right]dt$$

$$=\lim_{a\to 0}\frac{1}{2a}\int_{-a}^{a}f(t)dt=\lim_{a\to 0}\frac{2a\ f(a)}{2a}=f(0)$$

$$\int_{-\infty}^{\infty} \delta(\mathrm{t-c})\mathrm{f(t)dt} = \lim_{a\to 0}\frac{1}{2a}\int_{-\infty}^{\infty} f(t)\left[H(t-c+a)-H(t-c-a)\right]dt$$

$$= \lim_{a\to 0}\frac{1}{2a}\int_{-a+c}^{a+c} f(t)dt = \lim_{a\to 0}\frac{2a\ f(c)}{2a} = f(c)$$

$$F\{\delta(\mathrm{t})\} = \int_{-\infty}^{\infty}\delta(\mathrm{t})e^{-i\omega t}dt$$

$$= e^{-i\omega t}\big|_{t=0}$$

$$=1$$

$$F\{\delta(\mathrm{t-a})\} = \int_{-\infty}^{\infty}\delta(\mathrm{t-a})e^{-i\omega t}dt$$

$$= e^{-i\omega t}\big|_{t=a}$$

$$= e^{-i\omega a}$$

$$F\left\{\delta^{(\mathrm{n})}(\mathrm{t-a})\right\} = \int_{-\infty}^{\infty}\delta^{(n)}(\mathrm{t-a})\ e^{-i\omega t}dt$$

$$= (i\omega)^n F\{\delta(\mathrm{t-a})\}$$

$$= (i\omega)^n e^{-i\omega a}$$

$F\{F\{\delta(t)\}\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$ (By using the property of Theorem *Symmetry*)

(Note that delta function $\delta(t)$ is even➔ $\delta(t) = \delta(-t)$)

But $F\{F\{\delta(t)\}\} = F\{1\}$ ➔ $F\{1\} = 2\pi\delta(\omega)$