Fourier Sine Transform and Fourier Cosine Transform

Fourier sine transform: $$F_s\{f(t)\} = F(\omega) = \int_0^{\infty} f(t)\sin\omega t\,dt$$

Inverse Fourier sine transform: $$F_s^{-1}[F(\omega)] = f(t) = \frac{2}{\pi}\int_0^{\infty} F(\omega)\sin\omega t\,d\omega$$

Fourier cosine transform: $$F_c\{f(t)\} = F(\omega) = \int_0^{\infty} f(t)\cos\omega t\,dt$$

Inverse Fourier cosine transform: $$F_c^{-1}[F(\omega)] = f(t) = \frac{2}{\pi}\int_0^{\infty} F(\omega)\cos\omega t\,d\omega$$

$$\begin{aligned} F_c\{f'(t)\} &= \int_0^{\infty} f'(t)\cos\omega t\,dt \\ &= f(t)\cos\omega t\Big|_0^{\infty} - \int_0^{\infty} f(t)[-\omega\sin\omega t]\,dt \\ &= f(\infty)\cos\infty - f(0) + \omega\int_0^{\infty} f(t)\sin\omega t\,dt \\ &= \omega F_s\{f(t)\} - f(0) \end{aligned}$$

if we assume that $f(t)\to 0$ as $t\to\infty$

$$\begin{aligned} F_s\{f'(t)\} &= \int_0^{\infty} f'(t)\sin\omega t\,dt \\ &= f(t)\sin\omega t\Big|_0^{\infty} - \int_0^{\infty} f(t)[\omega\cos\omega t]\,dt \\ &= f(\infty)\sin\infty - \omega\int_0^{\infty} f(t)\cos\omega t\,dt \\ &= -\omega F_c\{f(t)\} \end{aligned}$$

if we assume that $f(t)\to 0$ as $t\to\infty$

➔ $$F_c\{f''(t)\} = -\omega^2 F_c\{f(t)\} - f'(0)$$
$$F_s\{f''(t)\} = -\omega^2 F_s\{f(t)\} + \omega f(0)$$

if we assume that $f(t)\to 0,\ f'(t)\to 0$ as $t\to\infty$

Example: Solve the boundary-value problem

$y''(t) - 9y(t) = 5e^{-2t} \quad 0 < t < \infty$

$y(0) = y_0$ and $y(\infty)$ bounded (two conditions for second order DE)

Due to the condition of $y(0) = y_0$

$$\left(\begin{array}{l} F_c\{f''(t)\} = -\omega^2 F_c\{f(t)\} - f'(0) \\ F_s\{f''(t)\} = -\omega^2 F_s\{f(t)\} + \omega f(0) \end{array} \right)$$

We take the Fourier sine transform of the ODE

$\Rightarrow$ $F_s\{y''(t) - 9y(t)\} = F_s\{5e^{-2t}\}, \qquad F_s\{y''(t)\} - 9F_s\{y(t)\} = 5F_s\{e^{-2t}\}$

$\Rightarrow$ $-\omega^2 F_s\{y(t)\} + \omega y_0 - 9F_s\{y(t)\} = 5\dfrac{\omega}{\omega^2+4}$

$\Rightarrow$ $-F_s\{y(t)\}(\omega^2+9) = 5\dfrac{\omega}{\omega^2+4} - \omega y_0, \qquad F_s\{y(t)\} = \dfrac{\omega y_0}{(\omega^2+9)} - 5\dfrac{\omega}{(\omega^2+4)(\omega^2+9)}$

$\Rightarrow$ $y(t) = F^{-1}\left\{\dfrac{\omega y_0}{(\omega^2+9)} - 5\dfrac{\omega}{(\omega^2+4)(\omega^2+9)}\right\}$

$$\frac{1}{(\omega^2+4)(\omega^2+9)} = \frac{1}{5}\frac{1}{(\omega^2+4)} - \frac{1}{5}\frac{1}{(\omega^2+9)}$$

$$\begin{aligned} y(t) &= F^{-1}\left\{\frac{\omega y_0}{(\omega^2+9)} - \frac{\omega}{(\omega^2+4)} + \frac{\omega}{(\omega^2+9)}\right\} \\ \Rightarrow \quad &= F^{-1}\left\{(y_0+1)\frac{\omega}{(\omega^2+9)} - \frac{\omega}{(\omega^2+4)}\right\} \\ &= (y_0+1)F^{-1}\left\{\frac{\omega}{(\omega^2+9)}\right\} - F^{-1}\left\{\frac{\omega}{(\omega^2+4)}\right\} \\ &= (y_0+1)e^{-3t} - e^{-2t} \end{aligned}$$

Note that $F_s\{e^{-at}\} = \int_0^\infty e^{-at} \sin \omega t \, dt = \dfrac{\omega}{a^2+\omega^2} \qquad (a > 0)$ $\Rightarrow$ $F_s^{-1}\left\{\dfrac{\omega}{a^2+\omega^2}\right\} = e^{-at}$

If $y'(0) = y_0'$ and $y(\infty)$ bounded are prescribed (two conditions for second order DE)

Then we will use the Fourier cosine transform to solve the ODE