複立葉積分與複立葉級數類似

如第 12 章所言，複立葉級數利用 $\left\{1,\ \cos\frac{n\pi x}{p},\ \sin\frac{n\pi x}{p}\right\}$ 這一組正交函數當基底(好像座標軸之基底)來逼近一週期性函數(週期 T=2P 為一有限值)或僅在一區間內定義之函數(如 $f(x)=x$, on [-P, P]，但可週期性複製於整個實數軸)。

The Fourier Series of a Function

Fourier series of $f(x)$ on $[-P,\ P]$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{P}+b_n\sin\frac{n\pi x}{P}\right]$$

where a_n, b_n are the Fourier coefficients

$$a_n=\frac{1}{P}\int_{-P}^{P} f(x)\ \cos\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n=0,\ 1,\ 2,...$$

$$b_n=\frac{1}{P}\int_{-P}^{P} f(x)\ \sin\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n=1,\ 2,\ 3,...$$

但是若面對一非週期性函數，如

$f(x)=x \quad for\ -1<x<1$， $f(x)=0 \quad for\ x\le -1,\ x\ge 1$. 注意這函數已完整定義於整個實數軸)，則仍可假設其具有一無窮大之週期 (因 P 無窮大，所以無返復性)，則級數累加展開之複立葉級數變成積分型式之複立葉積分。

由於上述複立葉積分與複立葉級數之相似性，所以複立葉積分與複立葉級數之收斂性質一致，即函數 f(x)若為一片斷平滑($f(x)$ and $f'(x)$ are piecewise continuous)，且 $\int_{-\infty}^{\infty}|f(x)dx|$ 收斂 (即函數瑕積分存在且值為有限，即可積分)，則複立葉積分收斂至 $\frac{1}{2}[f(x-)+f(x+)]$，即每一點之左(邊一點點)右(邊一點點)偏端點之平均值，當然，你只須利用收斂定理及原函數之定義(但若作奇函數或偶函數之 extension，則以 extension 後之奇函數或偶函數為依據)，即可寫出複立葉積分收斂值。

同樣，以複立葉積分表示非週期性函數時，同樣你有 $A(\omega)$, $B(\omega)$ 及複數型式之 $C(\omega)$，其中 $C(\omega)$ 即為複立葉轉換。

Fourier integral

If $f(x)$ and $f^{'}(x)$ are piecewise continuous in every finite interval

and $\int_{-\infty}^{\infty} f|(x)|dx$ converges, i.e $f(x)$ is absolutely integrable in $(-\infty, \ \infty)$.

➔ $f(x) = \frac{1}{\pi}\int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$

where $A(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x \ dx$, $B(\omega) = \int_{-\infty}^{\infty} f(x)\sin\omega x \ dx$

Or in **complex form** (複立葉積分之複數型式)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega x}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega) \ e^{i\omega x}d\omega$$

where $C(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

Convergence

$$\frac{1}{2}[f(x-) + f(x+)] = \frac{2}{\pi}\int_{0}^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$$

Or

$$\frac{1}{2}[f(x-) + f(x+)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega x}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega) \ e^{i\omega x}d\omega$$

Fourier cosine integral

The Fourier integral of an even function on the interval $(-\infty,\ \infty)$ is the cosine integral

$$f(x)=\frac{2}{\pi}\int_0^\infty A(\omega)\cos\omega x\, d\omega,\quad A(\omega)=\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$$

Fourier sine integral

The Fourier integral of an odd function on the interval $(-\infty,\ \infty)$ is the sine integral

$$f(x)=\frac{2}{\pi}\int_0^\infty B(\omega)\sin\omega x\, d\omega,\quad B(\omega)=\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$$

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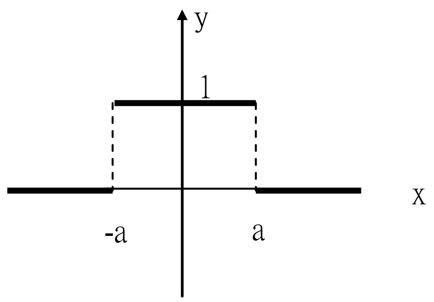
Example1:

$$f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$$



注意函數 $f(x)$ 已**完整**定義於**整個實數軸**，可假設其具有一無窮大之週期 (因 P 無窮大，所以無返復性)，則以積分型式之複立葉積分來代表函數 $f(x)$ 。

Fourier integral representation of the function

$$\Rightarrow f(x)=\frac{1}{\pi}\int_0^\infty [A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

$$A(\omega)=\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt=\int_{-a}^{a} f(t)\cos\omega t\, dt$$

$$=\int_{-a}^{a}\cos\omega t\, dt=\frac{\sin\omega t}{\omega}\Big|_{-a}^{a}=\frac{2\sin\omega a}{\omega}$$

$$B(\omega)=\int_{-\infty}^{\infty} f(t)\sin\omega t\ dt=\int_{-a}^{a} f(t)\sin\omega t\ dt$$

$$=\int_{-a}^{a} \sin\omega t\ dt=-\frac{\cos\omega t}{\omega}\Big|_{-a}^{a}=-\frac{\cos(a\omega)}{\omega}+\frac{\cos(-a\omega)}{\omega}=0$$

➔
$$f(x)=\frac{1}{\pi}\int_{0}^{\infty}\left[A(\omega)\cos\omega x+B(\omega)\sin\omega x\right]d\omega$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin a\omega\cos\omega x}{\omega}d\omega$$

注意:

由於函數 $f(x)$ 為一偶函數，所以可預期其 Fourier integral 即為 Fourier cosine integral
所以，若能一開始即判斷函數 $f(x)$ 為一偶函數，則可直接求取其 Fourier **cosine** integral
反之，若能一開始即判斷函數 $f(x)$ 為一奇函數，則可直接求取其 Fourier **sine** integral

Example2:

Represent $f(x)=\begin{cases}x^2 & for \quad 0\le x\le 10\\ 0 & for \quad x>10\end{cases}$ (1)by a cosine integral (2)by a sine integral

(1)By the definition of Fourier Cosine Integral

The Fourier cosine integral of f is $\int_{0}^{\infty} A(\omega)\cos(\omega x)d\omega$

$$A(\omega)=\frac{2}{\pi}\int_{0}^{\infty} f(t)\cos(\omega t)dt$$

$$=\frac{2}{\pi}\int_{0}^{10} t^2\cos(\omega t)dt=\frac{2}{\pi}\left[\frac{2t}{\omega^2}\cos(\omega t)+\left(\frac{t^2}{\omega}-\frac{2}{\omega^3}\right)\sin(\omega t)\right]\Bigg|_{0}^{10}$$

$$=\frac{2}{\pi}\left[\frac{20}{\omega^2}\cos(10\omega)+\left(\frac{100}{\omega}-\frac{2}{\omega^3}\right)\sin(10\omega)\right]$$

$$=\frac{4}{\pi\omega^3}\left[10\omega\cos(10\omega)+\left(50\omega^2-1\right)\sin(10\omega)\right]$$

The Fourier cosine integral of f converges to x^2 for $0\le x<10$, to 50 at $x=10$, to 0 for $x>10$

(2)By the definition of Fourier Sine Integral

The Fourier sine integral of f is $\int_0^\infty B(\omega)\sin(\omega x)d\omega$

$$B(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)dt$$

$$=\frac{2}{\pi}\int_0^{10} t^2\sin(\omega t)dt=\frac{2}{\pi}\left[\frac{2t}{\omega^2}\sin(\omega t)-\left(\frac{t^2}{\omega}-\frac{2}{\omega^3}\right)\cos(\omega t)\right]\Bigg|_0^{10}$$

$$=\frac{2}{\pi}\left[\frac{20}{\omega^2}\sin(10\omega)-\left(\frac{100}{\omega}-\frac{2}{\omega^3}\right)\cos(10\omega)-\frac{2}{\omega^3}\right]$$

$$=\frac{4}{\pi\omega^3}\left[10\omega\sin(10\omega)-\left(50\omega^2-1\right)\cos(10\omega)-1\right]$$

The Fourier sine integral of f converges to x^2 for $0\le x<10$, to 50 at $x=10$, to 0 for $x>10$

Example3：

Represent $f(x)=\begin{cases} k & for \quad 0\le x\le c \\ 0 & for \quad x>c \end{cases}$ (in which k is constant and c is positive constant.)

(1) by a cosine integral (2)by a sine integral

(1)The Fourier cosine integral of f is

$$\int_0^\infty A(\omega)\cos(\omega x)d\omega$$

with

$$A(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)dt$$

$$=\frac{2}{\pi}\int_0^{c} k\cos(\omega t)dt=\frac{2}{\pi}\left[\frac{k}{\omega}\sin(\omega t)\right]\Bigg|_0^{c}$$

$$=\frac{2k}{\pi\omega}\sin(c\omega)$$

The Fourier cosine integral of f converges to k for $0\le x<c$, to $k/2$ at $x=c$, to 0 for $x>c$

(2)The Fourier cosine integral of f is

$$\int_0^\infty B(\omega)\cos(\omega x)d\omega$$

with

$$B(\omega)=\frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)dt$$

$$=\frac{2}{\pi}\int_0^c k\sin(\omega t)dt=\frac{2}{\pi}\left[-\frac{k}{\omega}\cos(\omega t)\right]\Bigg|_0^c$$

$$=\frac{2k}{\pi\omega}\left[1-\cos(c\omega)\right]$$

The Fourier sine integral of f converges to to 0 at $x=0$, to k for $0<x<c$, to $k/2$ at $x=c$, to 0 for $x>c$

Fourier Transform Pairs

Fourier transform: $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

Inverse Fourier transform: $F^{-1}\left[F(\omega)\right]=f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega t}d\omega$