Existence of the Fourier series

Revised 2006.4.16

For the Fourier Series to exist, the Fourier coefficients must be finite. The (Weak) Dirichlet Condition guarantees this existence. It essentially says that the integral of the absolute value of the signal must be finite.

Definition: Fourier Coefficients and Series

Let $f(x)$ be Riemann integrable function on $[-P, \quad P]$

1. The numbers

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x) \cos\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n = 0, \quad 1, \quad 2, \ldots$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x) \sin\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n = 1, \quad 2, \quad 3, \ldots$$

are the Fourier coefficients of $f(x)$ on $[-P, \quad P]$.

2. The series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{P} + b_n \sin\frac{n\pi x}{P}\right]$$

is the Fourier series of $f(x)$ on $[-P, \quad P]$ when the constants are chosen to be the Fourier coefficients of $f(x)$ on $[-P, \quad P]$.

A function $f(x)$ is said to be absolutely integrable on $[a, \quad b]$ if

$\int_a^b |f(x)|dx < \infty$ ➔ exists and is finite

A function $f(x)$ is said to be square integrable on $[a, \quad b]$ if

$\int_a^b f^2(x)dx < \infty$ ➔ exists and is finite

It can be shown that each square integrable function on $[a, \quad b]$ is also absolutely integrable. If an integrable function is bounded on $[a, \quad b]$ then it is also square integrable. Further, if $f(x)$ and $g(x)$ two square integrable functions then the product $f(x)g(x)$ is absolutely integrable and therefore integrable.

➔ If $f(x)$ is square integrable on $[-P, \quad P]$, then

All of the coefficients $a_{n,} \quad b_n$ are definite numbers uniquely.

All of the coefficients $a_{n,} \quad b_n$ depend linearly on $f(x)$

$f(x)$ is called Riemann integrable on $[a, \; b]$, if there exists a number A having the following property:

> For every $\varepsilon > 0$ there exists a partition z_ε of $[a, \; b]$ such that for every partition Z finer than z_ε and for every choice of points ξ_k in $[x_{k-1}, x_k]$ we have $|S(f, Z, \xi) - A| < \varepsilon$.

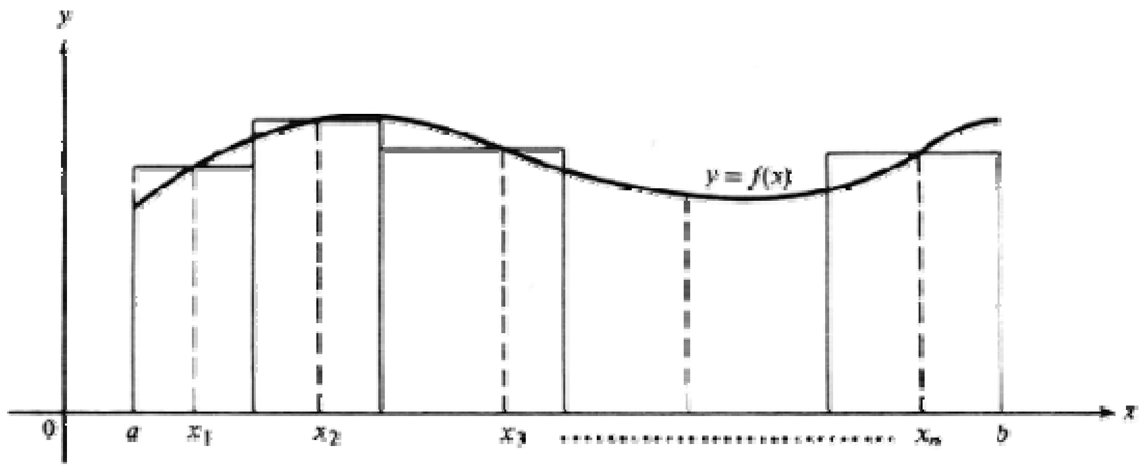


Figure A Riemann sum

Let $f : [a, \; b] \to R$. If f is continuous on [a,b], then f is Riemann integrable on [a,b].

Theorem
If *f* is Riemann integrable on [*a*,*b*], then *f* is bounded.

(http://mwt.e-technik.uni-ulm.de/world/lehre/basic_mathematics/di/node20.php3)

Convergence of Fourier Series

Peter Gustav Lejeune Dirichlet
(1805-1859)

德國數學家。在數論與分析、力學有卓越的貢獻，是解析數論與傅立葉級數的奠基者。他也是函數一詞的定義者。

It is important to establish simple criteria which determine when a Fourier series converges. Named after the German mathematician, Peter Dirichlet, the Dirichlet conditions are the sufficient conditions to guarantee existence and convergence of the Fourier series.

Definition (Piecewise Continuous Function)

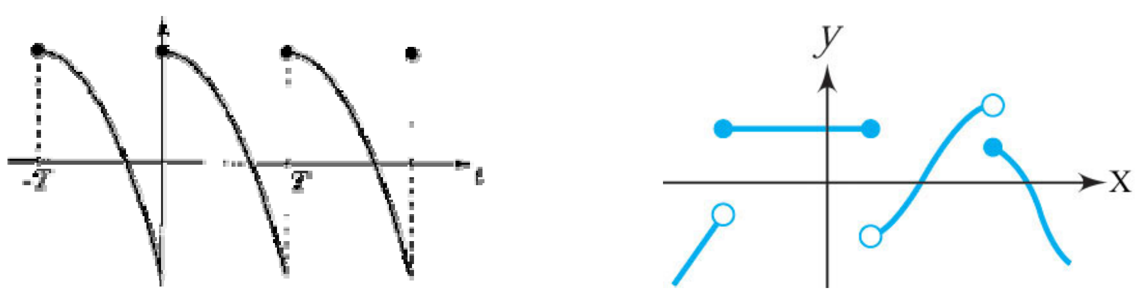
A function $f(x)$ defined on $I=[a,\ b]$ is said to be **piecewise continuous** on $[a,\ b]$ if and only if

1) there is a subdivision $a=x_0<x_1<x_2<...<x_n=b$ such that $f(x)$ is continuous on each subinterval $I_k=\{x: x_{k-1}<x<x_k\}$,

2) At each of the subdivision points $x_0, x_1, x_2, ..., x_n$ both one-sided limits of $f(x)$ exist.

➔ Both $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist and are finite.

➔ If x_0 is in $(a,\ b)$ and $f(x)$ is not continuous at x_0, then both $\lim_{x\to x_0+} f(x)$ and $\lim_{x\to x_0-} f(x)$ exist and are finite.



Typical piecewise continuous functions

Definition (Piecewise Smooth Function)

A function $f(x)$ is said to be piecewise smooth on $[a, \ b]$ if $f(x)$ and $f'(x)$ are piecewise continuous on $[a, \ b]$.

➔ A piecewise smooth function is therefore one that is continuous except possible for finitely many jump discontinuities and has a continuous derivative at all but finitely many point, where derivative may not exist but must have finite one-sided limits.

Condition of Dirichlet (http://www.nst.ing.tu-bs.de/schaukasten/fourier/en_idx.html)

The Fourier series of a periodic function x(t) exists, if

1. $\int_{T_0} |x(t)|\,dt < \infty$, i. e. x(t) is absolutely integratable,
2. variations of x(t) are limited in every finite time interval T and
3. there is only a finite set of discontinuities in T.

(http://cnx.rice.edu/content/m10089/latest/)

The (Weak) Dirichlet Condition for the Fourier Series

For the Fourier Series to exist, the Fourier coefficients must be finite. The Weak Dirichlet Condition guarantees this existence. It essentially says that the integral of the absolute value of the signal must be finite.

The (Strong) Dirichlet Conditions

For the Fourier Series to exist, the following two conditions must be satisfied (along with the Weak Dirichlet Condition):

1. In one period, $f(x)$ has only a finite number of minima and maxima.
2. In one period, $f(x)$ has only a finite number of discontinuities and each one is finite.

These are what we refer to as the Strong Dirichlet Conditions.

(http://www.mathphysics.com/pde/ch3wr.html)

If a function has a discontinuity, the Fourier series, when truncated to a large but finite number of terms, takes on a value between the right and left limits. The theorem says that the Fourier series finds the average of the two possibilities.

Theorem Suppose that $f(x)$ and $f'(x)$ are piecewise continuous on a finite interval $[a, \ b]$. Then the Fourier series converges at every value of x between a and b as follows:

The series converges to

1. $f(x)$ if x is a point of continuity.
2. $\frac{1}{2}[f(x+) + f(x-)]$ if x is a point of discontinuity.

At the end points, we have:

Then the series with coefficients converges to

$$\frac{1}{2}[f(a+) + f(b-)]$$

The limit at the end points is reasonable, because when the function is extended periodically, they are effectively the same point. And a particular consequence is that: *At places where such a function is continuous, the Fourier series does indeed converge to the function.*

In addition, if the function is continuous on the interval $[a, \ b]$, and $f(a) = f(b)$, then we can state a bit more, namely that the Fourier series converges *uniformly* to the function. This means that the error can be estimated independently of x:

Definition A sequence of functions $\{f_k(x)\}$ converges *uniformly* on the set Ω to a function $g(x)$ provided that

$$|f_k(x) - g(x)| < c_k$$

where c_k is a sequence of constants (independent of x in Ω) tending to 0.

The following theorem gives a general condition guaranteeing uniform convergence of Fourier series.

Theorem Suppose that $f'(x)$ is piecewise continuous, $f(x)$ itself is continuous on a finite interval $[a, \quad b]$, and $f(a) = f(b)$. Then

$$\max_{a \le x \le b} \left| a_0 + \sum_{m=1}^{M} a_m \cos\left(\frac{2\pi mx}{b-a}\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi nx}{b-a}\right) - f(x) \right| \to 0.$$

The condition that $f(a) = f(b)$ is again reasonable if you think of $f(x)$ as a periodic function extending beyond the interval $[a, \quad b]$ - the extended function would be discontinuous at the end points if $f(a)$ did not match $f(b)$.

---by R. A. Silverman

The Fourier series of a piecewise smooth (continuous or discontinuous) function $f(x)$ of period $2L$ converges for all values of x. The sum of the series equals $f(x)$ at every point of continuity and equals the number $\frac{1}{2}[f(x+) + f(x-)]$ the arithmetic mean of the right-hand and left-hand limits, at every point of discontinuity. If $f(x)$ is continuous everywhere, then the series converges absolutely and uniformly.