Partial Sums of Fourier Series

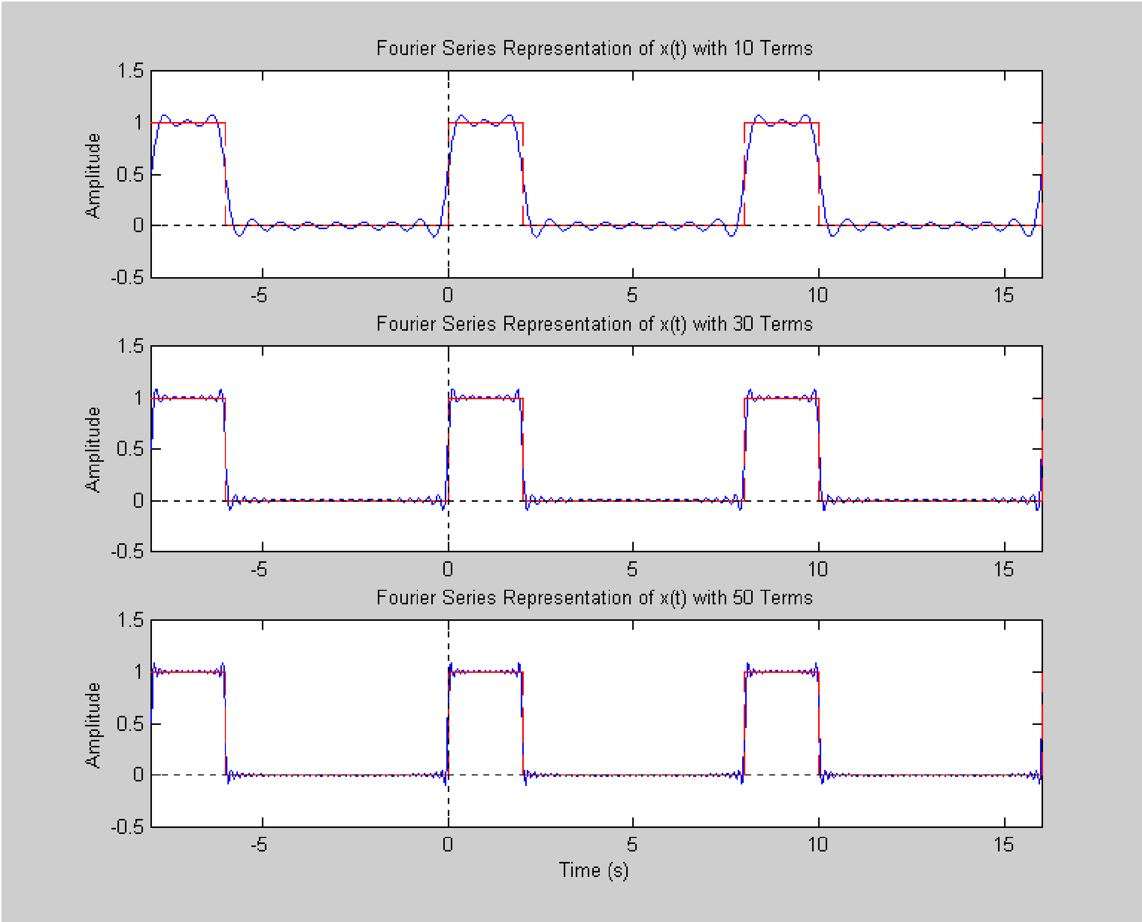
Revised 2006.4.16

(http://ece.gmu.edu/~gbeale/ece_220/fourier_series_02.html)

The complete Fourier series representation of a signal requires an infinite number of terms in general. When the complete series is used, the series converges to the exact value of the signal at every point where the signal is continuous and converges to the midpoint of the discontinuity wherever the signal is discontinuous.

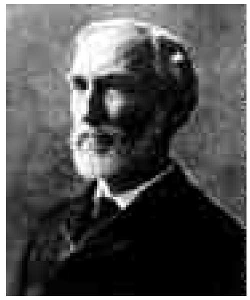
The signal $x(t)$ can be approximated by using a truncated form of the Fourier series, that is, stopping the summation after a finite number N of terms. The approximation may be 'good' or 'bad' in a subjective sense, but it will be the best approximation for a given N in terms of minimizing the mean squared error between the approximation and the actual signal. The approximation is given by

$$\hat{x}(t) = a_0 + \sum_{n=1}^{N} a_n \cos(n\omega_0 t) + \sum_{n=1}^{N} b_n \sin(n\omega_0 t), \quad N < \infty$$



The figure shows the approximations to the $x(t)$ in this example for $N = 10, 30, 50$. The figure shows clearly that as the number of terms increases, the approximation becomes better. The transitions between the two amplitude values become steeper and the magnitudes of the oscillations become smaller for the larger values of N. An interesting point to note in the graphs is that the magnitudes of the oscillations just before and just after the discontinuity do not decrease. In fact they increase slightly with our values of N from approximately 7.3% to 8.4% to 8.6% of the pulse height for the values of $N = 10, 30, 50$. The magnitude of the overshoot will eventually approach a value of 9% of the pulse height. This is known as the Gibbs phenomenon.

Gibbs Phenomenon



Josiah Willard Gibbs

(1839~ 1903)

J Willard Gibbs was an American mathematician best-known for the *Gibbs effect* seen when Fourier-analysing a discontinuous function.

(http://www-groups.dcs.st-and.ac.uk/~history/Posters2/Gibbs.html)

The non-uniform convergence of the Fourier series for discontinuous functions, and in particular the oscillatory behavior of the finite sum, was already analyzed by Wilbraham in 1848. This was later named the *Gibbs phenomenon*.

(http://lcavwww.epfl.ch/~prandoni/dsp/gibbs/gibbs.html)

A piecewise smooth function is a function defined over an interval $[a, \quad b]$ which is continuous along with its first derivative except for a finite number of first-order discontinuity points (jumps); the left and right limits for the function and its first derivative at the discontinuity points must exist. An example of this is the sawtooth function $f(x) = \pi, \quad \pi \in [-\pi, \quad \pi]$, , extended periodically on the real line; this function is discontinuous at $x = (2k+1)\pi$ for all interger values of k. Let us define $f_n(\pi)$ as the sum of the fist n Fourier series terms for the function $f(x)$:

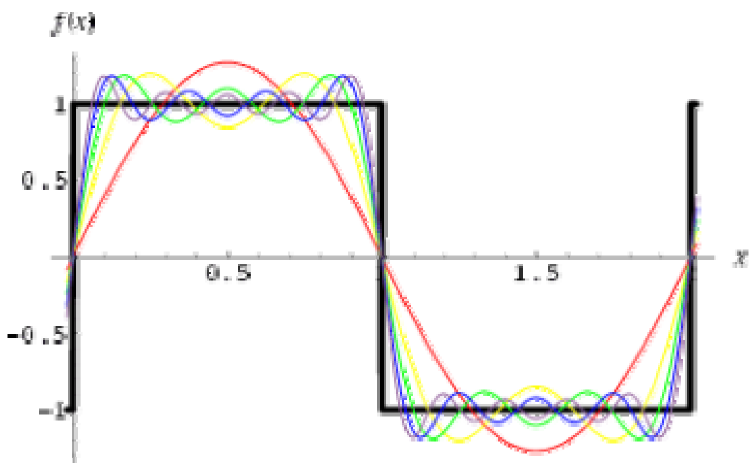
$$f_n(x) = \sum_{k=0}^{n} (-1)^{k+1} \frac{2}{k} \sin(kx)$$

It is noted the following facts:

- The Fourier series approximation displays an overshoot in the left-sided interval of the discontinuity (and a symmetric undershoot in the right-sided interval).
- While the convergence of the Fourier series to the sawtooth function improves anywhere else, the height of the overshoot does not decrease with augmenting the number of terms.

This behavior is called *Gibbs Phenomenon*. It shows that the Fourier series does not converge uniformly to a discontinuos function in an arbitrary small interval of the discontinuity point. Formally, the following theorem can be proven:

Interestingly, the overshoot factor depends only on the type of discontinuity and not on the values of the function.



The Gibbs phenomenon is an overshoot (or "ringing") of Fourier series occurring at simple discontinuities. The phenomenon is also illustrated above in the Fourier series of a square wave.

(http://mathworld.wolfram.com/GibbsPhenomenon.html)