複立葉轉換與複立葉積分

The Fourier Series of a Function

複立葉級數利用 $\left\{1,\ \cos\frac{n\pi x}{p},\ \sin\frac{n\pi x}{p}\right\}$ 這一組正交函數當基底(好像座標軸之基底)來逼近一週期性函數(週期 T=2P 為一有限值)或僅在一區間內定義之函數(如 $f(x)=x$, on [-P, P]，但可週期性複製於整個實數軸)。

Fourier series of $f(x)$ on $[-P,\ P]$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{P}+b_n\sin\frac{n\pi x}{P}\right]$$

where a_n, b_n are the Fourier coefficients

$$a_n=\frac{1}{P}\int_{-P}^{P} f(x)\ \cos\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for } n=0,\ 1,\ 2,\ldots$$

$$b_n=\frac{1}{P}\int_{-P}^{P} f(x)\ \sin\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for } n=1,\ 2,\ 3,\ldots$$

Complex Fourier Series

$$f(x)=\sum_{-\infty}^{\infty} c_n e^{in\pi x/p}$$

where $c_n=\frac{1}{2p}\int_{-p}^{p} f(x)e^{-in\pi x/p}dx, \qquad n=0,\ \pm 1\ ,\pm 2,\ldots\ldots$

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同樣，以複立葉積分表示非週期性函數時，同樣你有 $A(\omega)$, $B(\omega)$ 及複數型式之 $C(\omega)$，其中 $C(\omega)$ 即為複立葉轉換。

If $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval

and $\int_{-\infty}^{\infty} f|(x)|dx$ converges, i.e $f(x)$ is absolutely integrable in $(-\infty,\ \infty)$.

Fourier integral

➔ $f(x)=\frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos\omega x+B(\omega)\sin\omega x\right]d\omega$

where $A(\omega)=\int_{-\infty}^{\infty} f(x)\cos\omega x\ dx$, $B(\omega)=\int_{-\infty}^{\infty} f(x)\sin\omega x\ dx$

Or in **complex form** (複立葉積分之複數型式)

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega x}d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega)\ e^{i\omega x}d\omega$$

where $C(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

Fourier cosine integral

The Fourier integral of an even function on the interval $(-\infty,\ \infty)$ is the cosine integral

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty} A(\omega)\cos\omega x\ d\omega,\quad A(\omega)=\int_{-\infty}^{\infty} f(t)\cos\omega t\ dt$$

Fourier sine integral

The Fourier integral of an odd function on the interval $(-\infty,\ \infty)$ is the sine integral

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty} B(\omega)\sin\omega x\ d\omega,\quad B(\omega)=\int_{-\infty}^{\infty} f(t)\sin\omega t\ dt$$

Transform Pair:

首先注意到積分轉換是(正逆轉換)成雙成對的

If $f(x)$ is transformed into $F(\alpha)$ by an integral transform

$$F(\alpha)=\int_{a}^{b} f(x)K(\alpha,x)\ dx \longleftrightarrow f(x)=\int_{c}^{d} F(\alpha)H(\alpha,x)\ d\alpha$$

K, H ➔ *kernels*

同上，我們有 **Fourier Transform Pairs 的定義，分別與複立葉積分(複數型式)及複立葉 cosine 積分、複立葉 sine 積分相關**

Fourier Transform Pairs

Fourier transform: $F(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

Inverse Fourier transform: $F^{-1}[F(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$

請對照 Fourier Transform & Inverse Fourier transform 與 **complex Fourier integral 的關係**

complex Fourier integral (複立葉積分之複數型式)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\right)e^{i\omega x}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\omega)\, e^{i\omega x}d\omega \quad \text{where} \quad C(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

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even functions

Fourier cosine transform: $F_c\{f(t)\} = F(\omega) = \int_{0}^{\infty} f(t)\cos\omega t\ dt$

Inverse Fourier cosine transform: $F_c^{-1}[F(\omega)] = f(t) = \frac{2}{\pi}\int_{0}^{\infty} F(\omega)\cos\omega t\ d\omega$

請對照 Fourier cosine transform & Inverse Fourier cosine transform 與 Fourier cosine integral **的關係**

Fourier cosine integral

The Fourier integral of an even function on the interval $(-\infty, \ \infty)$ is the cosine integral

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} A(\omega)\cos\omega x\ d\omega, \quad A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\ dt$$

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odd functions

Fourier sine transform: $F_s\{f(t)\} = F(\omega) = \int_{0}^{\infty} f(t)\sin\omega t\ dt$

Inverse Fourier sine transform: $F_s^{-1}[F(\omega)] = f(t) = \frac{2}{\pi}\int_{0}^{\infty} F(\omega)\sin\omega t\ d\omega$

請對照 Fourier cosine transform & Inverse Fourier sine transform 與 Fourier sine integral **的關係**

Fourier sine integral

The Fourier integral of an odd function on the interval $(-\infty, \ \infty)$ is the sine integral

$$f(x) = \frac{2}{\pi}\int_0^{\infty} B(\omega)\sin\omega x \, d\omega, \quad B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t \ dt$$

Example1 (complex Fourier integral vs. Fourier transform)

$f(x) = e^{-a|x|}$ for all real x, with a a positive constant.

$$\Rightarrow f(x) = \begin{cases} e^{-ax} & \text{for } x \geq 0 \\ e^{ax} & \text{for } x < 0 \end{cases}$$

$$C_{\omega} = \int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}dt = \int_{-\infty}^{0} e^{at}e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt$$

$$= \int_{-\infty}^{0} e^{(a-i\omega)t}dt + \int_{0}^{\infty} e^{-(a+i\omega)t}dt$$

$$= \frac{e^{(a-i\omega)t}}{a-i\omega}\Big|_{-\infty}^{0} - \frac{e^{-(a+i\omega)t}}{(a+i\omega)}\Big|_{0}^{\infty}$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}$$

➔ The complex Fourier integral of $f(x) = e^{-a|x|}$

$$\Rightarrow \frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{\omega}e^{i\omega x}d\omega = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}e^{i\omega x}d\omega$$

Further, the function $f(x) = e^{-a|x|}$ is continuous for all x. So the complex Fourier integral converges to $f(x)$ for all x.

In other words, the Fourier transform of $f(x) = e^{-a|x|}$

$$\Rightarrow F\left\{e^{-a|t|}\right\} = \int_{-\infty}^{\infty} e^{-a|t|}e^{-i\omega t}dt = \frac{2a}{a^2+\omega^2}$$

Example 2 (complex Fourier integral vs. Fourier transform)

$$f(x) = xe^{-|x|}$$

$$C_\omega = \int_{-\infty}^{\infty} xe^{-|x|}e^{-i\omega x}dx = \int_{-\infty}^{0} xe^{x}e^{-i\omega x}dx + \int_{0}^{\infty} xe^{-x}e^{-i\omega x}dx$$

$$= x\frac{e^{(1-i\omega)x}}{1-i\omega}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{e^{(1-i\omega)x}}{1-i\omega}dx + x\frac{e^{-(1+i\omega)x}}{-(1+i\omega)}\Big|_{0}^{\infty} - \int_{0}^{\infty} \frac{e^{-(1+i\omega)x}}{-(1+i\omega)}dx$$

$$= -\frac{e^{(1-i\omega)x}}{(1-i\omega)^2}\Big|_{-\infty}^{0} - \frac{e^{-(1+i\omega)x}}{(1+i\omega)^2}\Big|_{0}^{\infty}$$

$$= -\frac{1}{(1-i\omega)^2} + \frac{1}{(1+i\omega)^2} = \frac{-4i\omega}{(1+\omega^2)^2}$$

➔ The complex Fourier integral of $f(x) = xe^{-|x|}$

➔ $$\frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_\omega e^{i\omega x}d\omega = -\frac{2i}{\pi}\int_{-\infty}^{\infty} \frac{\omega}{(1+\omega^2)^2}e^{i\omega x}d\omega$$

Further, the function $f(x) = xe^{-|x|}$ is continuous for all x. So the complex Fourier integral converges to $f(x)$ for all x.

In other words, the Fourier transform of $f(x) = xe^{-|x|}$

➔ $$F\{te^{-|t|}\} = \int_{-\infty}^{\infty} te^{-|t|}e^{-i\omega t}dt = \frac{-4i\omega}{(1+\omega^2)^2}$$

Example 3

$$f(x) = 5[H(x-3) - H(x-11)]$$

$$F\{f(t)\} = \int_{-\infty}^{\infty} 5[H(t-3) - H(t-11)]e^{-i\omega t}dt$$

$$= \int_{3}^{\infty} 5e^{-i\omega t}dt - \int_{11}^{\infty} 5e^{-i\omega t}dt = \frac{5e^{-i\omega t}}{-i\omega}\Big|_{3}^{\infty} - \frac{5e^{-i\omega t}}{-i\omega}\Big|_{11}^{\infty}$$

$$= \frac{5e^{-3i\omega}}{i\omega} - \frac{5e^{-11i\omega}}{i\omega} = 5\frac{e^{-3i\omega} - e^{-11i\omega}}{i\omega}$$

$$= 5e^{-7i\omega}\frac{e^{4i\omega} - e^{-4i\omega}}{i\omega} = 5e^{-7i\omega}\frac{2i\sin(4\omega)}{i\omega} = 10e^{-7i\omega}\frac{\sin(4\omega)}{\omega}$$

Example 4

$f(x) = e^{-ax}$ for all real x, with a a positive constant.

$$F\{e^{-at}\} = \int_{-\infty}^{\infty} e^{-at} e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-(a+i\omega)t} dt = \int_{-\infty}^{0} e^{-(a+i\omega)t} dt + \int_{0}^{\infty} e^{-(a+i\omega)t} dt$$

$$= -\frac{e^{-(a+i\omega)t}}{(a+i\omega)}\Big|_{-\infty}^{0} - \frac{e^{-(a+i\omega)t}}{(a+i\omega)}\Big|_{0}^{\infty}$$

$$= -\frac{1}{a+i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}$$

➔ The complex Fourier integral of $f(x) = e^{-a|x|}$

$$➔ \quad \frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_\omega e^{i\omega x} d\omega = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2} e^{i\omega x} d\omega$$

Further, the function $f(x) = e^{-a|x|}$ is continuous for all x. So the complex Fourier integral converges to $f(x)$ for all x.

In other words, the Fourier transform of $f(x) = e^{-a|x|}$

$$➔ \quad F\{e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\omega t} dt = \frac{2a}{a^2+\omega^2}$$