Fourier Series

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Joseph Fourier

(1768 ~ 1830)

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

(http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html)

Fourier series, named in honor of Joseph Fourier (1768-1830), is a representation of a periodic function as an infinite series of constants multiplied by sine and/or cosine functions of different frequencies.

A great thing about using Fourier series on periodic functions is that the first few terms often are a pretty good approximation to the whole function, not just the region around a special point.

Fourier series are used extensively in engineering, especially for processing images and other signals.

Definition：Periodic functions

A function $f(x)$ is said to have a **period** T or to be **periodic** with period T if for all x, $f(x+T)=f(x)$, where P is a positive constant. The least value of $T>0$ is called the **fundamental period** or the **principal period** or simply the **period** of $f(x)$.

<u>Example 1</u>

The function $\cos(x)$ has periods $2\pi,\quad 4\pi,\quad 6\pi,\ldots$, since $\cos(x+2\pi),\quad \cos(x+4\pi),\quad \cos(x+6\pi),\ldots$ all equal $\cos(x)$.

<u>Example 2</u>

Find the **fundamental period** of $\cos\left(\frac{n\pi x}{P}\right)$

First, we assume the **fundamental period** of $\cos\left(\frac{n\pi x}{P}\right)$ is T

By definition, we have $\cos\left(\frac{n\pi x}{P}\right)=\cos\left(\frac{n\pi(x+T)}{P}\right)$ (Note： $x\to x+T$)

➔ $\cos\left(\frac{n\pi(x+T)}{L}\right)=\cos\left(\frac{n\pi x}{P}+\frac{n\pi T}{P}\right)$ ➔ $\frac{n\pi T}{P}=2n\pi$ ➔ $T=2P$

<u>Example 3</u>

If $f(x)$ has the period *2P*, then $\int_{-P}^{P} f(x)dx=\int_{a-P}^{a+P} f(x)dx$ for all real a

The Fourier Series of a Function

Fourier series of $f(x)$ on $[-P, \quad P]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{P} + b_n \sin\frac{n\pi x}{P} \right]$$

where a_n, b_n are the Fourier coefficients.

To compute the Fourier coefficients a_n, b_n, we need the following integral identities:

1) $\int_{-P}^{P} \cos\left(\frac{n\pi x}{P}\right)\sin\left(\frac{n\pi x}{P}\right)dx = 0$

2) If $n \neq m$, then

$$\int_{-P}^{P} \cos\left(\frac{n\pi x}{P}\right)\cos\left(\frac{m\pi x}{P}\right)dx = \int_{-P}^{P} \sin\left(\frac{n\pi x}{P}\right)\sin\left(\frac{m\pi x}{P}\right)dx0$$

3) If $n \neq 0$, then

$$\int_{-P}^{P} \cos^2\left(\frac{n\pi x}{P}\right)dx = \int_{-P}^{P} \sin^2\left(\frac{n\pi x}{P}\right)dx = P$$

4) $\int_{-P}^{P} \cos\left(\frac{n\pi x}{P}\right)dx = 0$

5) $\int_{-P}^{P} \sin\left(\frac{n\pi x}{P}\right)dx = 0$

Of course, you can check the above integral identities using the orthogonality relations of the trigonometric function (as mentioned in class) or based on the concept of even and odd functions. However, you also can check the above integral identities by operations of trigonometric functions as follows:

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\ \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\ \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\ \sin\left(\frac{B-A}{2}\right)$$

$$\sin A\ \cos B = [\sin(A-B) + \sin(A+B)]/2$$

or $$\cos A\ \cos B = [\cos(A-B) + \cos(A+B)]/2$$

$$\sin A\ \sin B = [\cos(A-B) - \cos(A+B)]/2$$

Therefore, Fourier series of $f(x)$ on $[-P,\ \ P]$ can be obtained as in the following:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{P} + b_n \sin\frac{n\pi x}{P} \right]$$

$$a_n = \frac{1}{P}\int_{-P}^{P} f(x)\ \cos\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n = 0,\quad 1,\quad 2,\ldots$$

$$b_n = \frac{1}{P}\int_{-P}^{P} f(x)\ \sin\left(\tfrac{n\pi x}{P}\right)dx \quad \text{for} \quad n = 1,\quad 2,\quad 3,\ldots$$

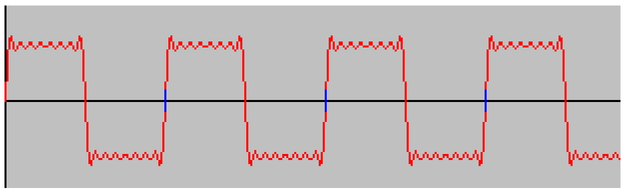
Note that, we do not have any x polynomial term in the Fourier series. We only have a constant term $\frac{a_0}{2}$ and trigonometric polynomials in the Fourier series.

Some examples of Fourier series expansion

Square wave

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ h, & 0 < x < \pi \end{cases}$$

➔ $$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)$$

➔ 

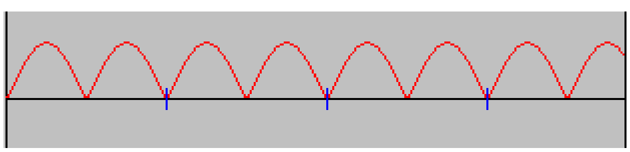
for n_{max}=15

(http://www.phy.ntnu.edu.tw/java/sound/Fourier.html)

Full wave rectifier

$$f(t) = \begin{cases} \sin \omega t, & 0 < \omega t < \pi, \\ -\sin \omega t, & -\pi < \omega t < 0. \end{cases}$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}.$$



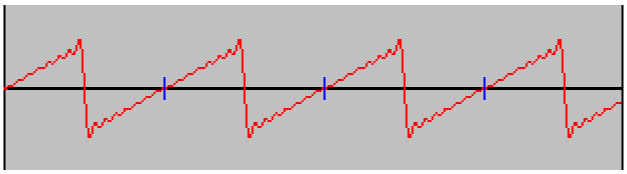
for n_{max}=14

(http://www.phy.ntnu.edu.tw/java/sound/Fourier.html)

Swath wave

$$f(x) = \begin{cases} x, & 0 \le x \le \pi, \\ x-2\pi, & \pi < x \le 2\pi \end{cases}$$

$$f(x) = x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ... + (-1)^{n+1}\frac{\sin nx}{n} + ...\right]$$



for n_{max}=15

(http://www.phy.ntnu.edu.tw/java/sound/Fourier.html)

Up to now, we have not confirmed the existence and convergence of Fourier series yet. Is it a convergent series ? Does it converge to the function $f(x)$ or a different function ? Answer to these questions, we need to learn about the convergence of Fourier series.