HOMEWORK #2s (9.2 ~9.5)

Due on March 8

1) Suppose $\vec{r}(t)=t^2\vec{i}+(t^3-2t)\vec{j}+(t^2-5t)\vec{k}$ is the position vector of a moving particle. At what points does the particle pass through the xy-plane ? What are its speed, velocity, acceleration and tangent line (to the curve traced by $\vec{r}(t)$) at these points ? (Problem 9, page 457).

ANS The particle passes through the xy-plane when $z(t)=t^2-5t=0 \; or \; t=0,5$ which gives us the points (0,0,0) and (25,115,0).

$$\vec{v}(t)=2t\vec{i}+(3t^2-2)\vec{j}+(2t-5)\vec{k};\ \vec{v}(0)=-2\vec{j}-5\vec{k},\ \vec{v}(5)=10\vec{i}+73\vec{j}+5\vec{k};$$
$$\vec{a}(t)=2\vec{i}+6t\vec{j}+2\vec{k};\ \vec{a}(0)=2\vec{i}+2\vec{k},\ \vec{a}(5)=2\vec{i}+30\vec{j}+2\vec{k}$$

tangent line through $(0,0,0)$: $x=0+0t,\ y=0+(-2)t,\ z=0+(-5)t$

tangent line through $(25,115,0)$: $x=25+10t,\ y=115+73t,\ z=0+5t$

speed: $\left|\vec{v}(0)\right|=\sqrt{(-2)^2+(-5)^2}=\sqrt{29}$, $\left|\vec{v}(5)\right|=\sqrt{10^2+73^2+5^2}=\sqrt{5454}$

2) Suppose $\vec{r}(t)$ is the position vector of a moving particle. Find the curvature, the tangential and normal components of the acceleration at any t. $\vec{r}(t)=5\cos(t)\vec{i}+5\sin(t)\vec{j}$ (Problem 13, page 463).

ANS $\vec{v}(t)=-5\sin t\vec{i}+5\cos t\vec{j},\ \left|\vec{v}(t)\right|=5;\ \vec{a}(t)=-5\cos t\vec{i}-5\sin t\vec{j};\ \vec{v}\cdot\vec{a}=0,\ \vec{v}\times\vec{a}=25\vec{k}$,

$$\left|\vec{v}\times\vec{a}\right|=25;\ a_T=0,\ a_N=5$$

$$\kappa=\frac{\left|\vec{v}\times\vec{a}\right|}{\left|\vec{v}\right|^3}=\frac{25}{5^3}=\frac{1}{5}$$

3) If $u=f(x,y)$ and $x=r\cos\theta,\ y=r\sin\theta$, show that Laplace's equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$ becomes $\dfrac{\partial^2u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2u}{\partial\theta^2}=0$ (Problem 53, page 470).

ANS With $x=r\cos\theta \; and \; y=r\sin\theta$

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial^2u}{\partial r^2}=\frac{\partial^2u}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta+\frac{\partial^2u}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta=\frac{\partial^2u}{\partial x^2}\cos^2\theta+\frac{\partial^2u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \theta}(-r\sin\theta) + \frac{\partial u}{\partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \theta}(r\cos\theta)$$

$$= -r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta.$$

Using $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$ we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + \frac{1}{r}(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta)$$

$$+\frac{1}{r^2}(-r\frac{\partial u}{\partial x}\cos\theta + r^2\frac{\partial^2 u}{\partial x^2}\sin^2\theta - r\frac{\partial u}{\partial y}\sin\theta + r^2\frac{\partial^2 u}{\partial y^2}\cos^2\theta)$$

$$= \frac{\partial^2 u}{\partial x^2}(\cos^2\theta + \sin^2\theta) + \frac{\partial^2 u}{\partial y^2}(\sin^2\theta + \cos^2\theta) + \frac{\partial u}{\partial x}(\frac{1}{r}\cos\theta - \frac{1}{r}\cos\theta) + \frac{\partial u}{\partial y}(\frac{1}{r}\sin\theta - \frac{1}{r}\sin\theta)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

4) Find the directional derivative of the given function at the given point in the directed direction $f(x,y) = \tan^{-1}\frac{y}{x}; (2,-2)$, $\vec{i} - 3\vec{j}$ (Problem 13, page 475).

ANS $\vec{u} = \frac{\sqrt{10}}{10}\vec{i} - \frac{3\sqrt{10}}{10}\vec{j}; \nabla f = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}; \nabla f(2,-2) = \frac{1}{4}\vec{i} + \frac{1}{4}\vec{j}$

$$D_{\vec{u}} f(2,-2) = \frac{\sqrt{10}}{40} - \frac{3\sqrt{10}}{40} = -\frac{\sqrt{10}}{20}$$