HOMEWORK #4s(9.9 ~9.12)

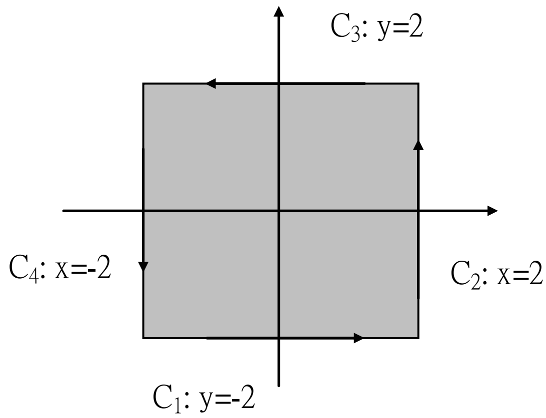
Due on March 22

1) Consider the line integral in Example 4 (page 518) on the curve C consisting of the four straight segments C_1, C_2, C_3, C_4 $\oint_c \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$.

(1) Is Green's theorem applicable to the line integral ? (i.e. Can you relate the line integral to a double integral using Green's theorem) ? Why ?

(2) Compute the line integral on the curve C by using $\oint_c = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4}$

ANS (1) Green's theorem is not applicable since P, Q, $\frac{\partial P}{\partial y}$, and $\frac{\partial Q}{\partial x}$ are not continuous at the origin.



(2) $\oint_c \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \to \oint_c \vec{F}\cdot d\vec{r}$

$\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$

$c = c_1 + c_2 + c_3 + c_4$

$c_1 : \vec{r}(t) = (-2+t)\vec{i} - 2\vec{j}\,,\ \ d\vec{r} = \vec{i}dt\,,\ \ t = 0 \sim 4;\ \ \vec{F} = \frac{2}{(-2+t)^2+4}\vec{i} + \frac{-2+t}{(-2+t)^2+4}\vec{j}$

$c_2 : \vec{r}(t) = 2\vec{i} + (-2+t)\vec{j}\,,\ \ d\vec{r} = \vec{j}dt\,,\ \ t = 0 \sim 4;\ \ \vec{F} = \frac{2-t}{(-2+t)^2+4}\vec{i} + \frac{2}{(-2+t)^2+4}\vec{j}$

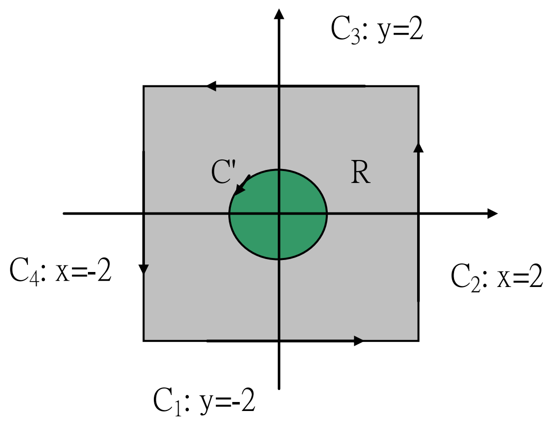
$c_3 : \vec{r}(t) = (2-t)\vec{i} + 2\vec{j}\,,\ \ d\vec{r} = -\vec{i}dt\,,\ \ t = 0 \sim 4;\ \ \vec{F} = \frac{-2}{(2-t)^2+4}\vec{i} + \frac{2-t}{(2-t)^2+4}\vec{j}$

$c_4 : \vec{r}(t) = -2\vec{i} + (2-t)\vec{j}\,,\ \ d\vec{r} = -\vec{j}dt\,,\ \ t = 0 \sim 4;\ \ \vec{F} = \frac{t-2}{(2-t)^2+4}\vec{i} + \frac{-2}{(2-t)^2+4}\vec{j}$

$\oint_c \vec{F}\cdot d\vec{r} = \int_{c_1} + \int_{c_2} + \int_{c_3} + \int_{c_4} \vec{F}\cdot d\vec{r} = 2\int_0^4 \frac{2}{(-2+t)^2+4}dt + 2\int_0^4 \frac{2}{(2-t)^2+4}dt = 2\pi$

2) Consider the line integral in Example 6 (page 519) on the curve C consisting of the four straight segments C_1, C_2, C_3, C_4. Apply the Green's theorem to evaluate line integral on the curve C.

Note that we choose the curve C' to be $x^2 + y^2 = a^2$ where a is small enough so that the circle lies entirely with C.



ANS $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2+y^2)^2} = \dfrac{\partial Q}{\partial x}$

$$\to \oint_c \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \oint_{c'} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$x = a\cos\theta \quad dx = -a\sin\theta d\theta$
$y = a\sin\theta \quad dy = a\cos\theta d\theta$

$$\oint_{c'} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} \frac{-a\sin\theta}{a^2}(-a\sin\theta d\theta) + \frac{a\cos\theta}{a^2}(a\cos\theta d\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)d\theta = 2\pi$$

3) A lamina has the shape of the region bounded by the graph of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

If its density is $\rho(x,y) = 1$, find (Problem 61, page 510).

(1) the moment of inertia about the x-axis of the lamina,

(2) the moment of inertia about the y-axis of the lamina,

(3) the radius of gyration about the x-axis,

(4) the radius of gyration about the y-axis

ANS (1) $I_x = \iint_R y^2\rho(x,y)dA = \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dydx = \dfrac{2}{3}\dfrac{b^3}{a^3}\int_{-a}^{a}(a^2-x^2)^{\frac{3}{2}}dx = \dfrac{2}{3}ab^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta$

$$= \frac{2}{3}ab^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}(1+\cos 2\theta)^2 d\theta = \frac{1}{6}ab^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2\theta + \frac{1}{2} + \frac{1}{2}\cos 4\theta)d\theta$$

$x = a\sin\theta$
$dx = a\cos\theta d\theta$

$$= \frac{1}{6}ab^3(\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{ab^3\pi}{4}$$

(2) $I_y = \iint_R x^2\rho(x,y)dA = \int_{-a}^{a}\int_{\frac{-b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 dydx = \dfrac{2b}{a}\int_{-a}^{a}x^2(a^2-x^2)^{\frac{1}{2}}dx$

$x = a\sin\theta$
$dx = a\cos\theta d\theta$

$$= 2a^3b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta d\theta = 2a^3b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}(1-\cos^2 2\theta)d\theta$$

$$= \frac{1}{2}a^3b\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\frac{1}{2}-\frac{1}{2}\cos 4\theta)d\theta = \frac{1}{2}a^3b(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{a^3b\pi}{4}$$

(3) Using $m = \pi ab$

$$, \quad R_g = \sqrt{\frac{I_x}{m}} = \frac{1}{2}\sqrt{\frac{ab^3\pi}{\pi ab}} = \frac{1}{2}b$$

$$(4)\, R_g = \sqrt{\frac{I_y}{m}} = \frac{1}{2}\sqrt{\frac{a^3b\pi}{\pi ab}} = \frac{1}{2}a$$

4) Find the center of mass of the lamina that has the given shape and density

$y = \sqrt{3}x$, $y = 0$, $x = 3$; $\rho(r,\theta) = r^2$ (Problem 13, page 514).

ANS $x = 3 \to r\cos\theta = 3 \to r = 3\sec\theta$

$$y = \sqrt{3}x \to \theta = \frac{\pi}{3}$$

$$m = \int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} r^2 r dr d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{4} r^4 \Big|_0^{3\sec\theta} d\theta = \frac{81}{4}\int_0^{\frac{\pi}{3}} \sec^4\theta d\theta$$

$$= \frac{81}{4}\int_0^{\frac{\pi}{3}} (1+\tan^2\theta)\sec^2\theta d\theta = \frac{81}{4}(\tan\theta + \frac{1}{3}\tan^3\theta)\Big|_0^{\frac{\pi}{3}} = \frac{81}{2}\sqrt{3}$$

$$M_y = \int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} x r^2 r dr d\theta = \int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} r^4 \cos\theta dr d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{5} r^5 \cos\theta \Big|_0^{3\sec\theta} d\theta$$

$$= \frac{243}{5}\int_0^{\frac{\pi}{3}} \sec^5\theta \cos\theta d\theta = \frac{243}{5}\int_0^{\frac{\pi}{3}} \sec^4\theta d\theta = \frac{486}{5}\sqrt{3}$$

$$M_x = \int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} y r^2 r dr d\theta = \int_0^{\frac{\pi}{3}}\int_0^{3\sec\theta} r^4 \sin\theta dr d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{5} r^5 \sin\theta \Big|_0^{3\sec\theta} d\theta$$

$$= \frac{243}{5}\int_0^{\frac{\pi}{3}} \sec^5\theta \sin\theta d\theta = \frac{243}{5}\int_0^{\frac{\pi}{3}} \tan\theta \sec^4\theta d\theta$$

$$= \frac{243}{5}\int_0^{\frac{\pi}{3}} \tan\theta(1+\tan^2\theta)\sec^2\theta d\theta$$

$$= \frac{243}{5}\int_0^{\frac{\pi}{3}} (\tan\theta + \tan^3\theta)\sec^2\theta d\theta = \frac{243}{5}(\frac{1}{2}\tan^2\theta + \frac{1}{4}\tan^4\theta)\Big|_0^{\frac{\pi}{3}} = \frac{729}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{486\sqrt{3}/5}{81\sqrt{3}/2} = \frac{12}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{729/4}{81\sqrt{3}/2} = \frac{3\sqrt{3}}{2}$$

The center of mass is $(\frac{12}{5}, \frac{3\sqrt{3}}{2})$

5) Show that the given integral is independent of the path and the evaluate it

$$\int_{(1,0,0)}^{(2,\pi/2,1)} (2x\sin y + e^{3z})dx + x^2\cos y dy + (3xe^{3z}+5)dz$$ (Problem 21, page 502).

ANS $\int_c Pdx + Qdy + Rdz$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, and $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ is independent of the path C

$$\int_{(1,0,0)}^{(2,\frac{\pi}{2},1)} (2x\sin y + e^{3z})dx + x^2\cos y dy + (3xe^{3z}+5)dz$$

$\frac{\partial P}{\partial y} = 2x\cos y = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = 3e^{3z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y} \rightarrow$ the integral is independent of path

$$\frac{\partial \phi}{\partial x} = 2x\sin y + e^{3z} \rightarrow \phi = x^2\sin y + xe^{3z} + f(y,z)$$

$$\frac{\partial \phi}{\partial y} = x^2\cos y \rightarrow \phi = x^2\sin y + g(x,z)$$

$$\frac{\partial \phi}{\partial z} = 3xe^{3z} + 5 \rightarrow \phi = xe^{3z} + 5z + h(x,y)$$

$$\therefore \phi(x,y,z) = x^2\sin y + xe^{3z} + 5z$$

$$\int_{(1,0,0)}^{(2,\frac{\pi}{2},1)} (2x\sin y + e^{3z})dx + x^2\cos y dy + (3xe^{3z}+5)dz$$

$$= \int_{(1,0,1)}^{(2,\frac{\pi}{2},1)} d(x^2\sin y + xe^{3z} + 5z)$$

$$= (x^2\sin y + xe^{3z} + 5z)\Big|_{(1,0,0)}^{(2,\frac{\pi}{2},1)} = [4(1) + 2e^3 + 5] - [0 + 1 + 0]$$

$$= 8 + 2e^3$$