HOMEWORK #5s(9.12 ~9.16)

Due on March 29

1) Proceed as in Example 6 (page 519) to evaluate the given line integral $\oint_c \frac{-y^3dx+xy^2dy}{(x^2+y^2)^2}$ where C is the ellipse $x^2+4y^2=4$ (Problem 25, page 521).

ANS We first observe that $P_y=(y^4-3x^2y^2)/(x^2+y^2)^3=Q_x$ Letting C' be the circle

$$x^2+y^2=\frac{1}{4}$$

$$x=\frac{1}{4}\cos t,\ dx=-\frac{1}{4}\sin tdt$$
$$y=\frac{1}{4}\sin t,\ dy=\frac{1}{4}\cos tdt$$

we have $\oint_C \frac{-y^3dx+xy^2dy}{(x^2+y^2)^2}=\oint_{C'} \frac{-y^3dx+xy^2dy}{(x^2+y^2)^2}$

$$=\int_0^{2\pi}\frac{-\frac{1}{64}\sin^3 t(-\frac{1}{4}\sin tdt)+\frac{1}{4}\cos t(\frac{1}{16}\sin^2 t)(\frac{1}{4}\cos tdt)}{1/256}$$

$$=\int_0^{2\pi}(\sin^4 t+\sin^2 t\cos^2 t)dt=\int_0^{2\pi}(\sin^4 t+(\sin^2 t-\sin^4 t)dt$$

$$=\int_0^{2\pi}\sin^2 tdt=(\frac{1}{2}t-\frac{1}{4}\sin 2t)\Big|_0^{2\pi}=\pi$$

2) In the problem (Problem 25, page 528), evaluate $\iint_S (3z^2+4yz)dS$, where S is the portion of the plane $x+2y+3z=6$ in the first octant. Use the portion of S onto the coordinate plane indicated in the given plane.

ANS Write the equation of the surface as $y=\frac{1}{2}(6-x-3z)$

$$y_x=-\frac{1}{2},\quad y_z=-\frac{3}{2};\quad dS=\sqrt{1+1/4+9/4}\,dA=\frac{\sqrt{14}}{2}dA$$

$$\iint_S(3z^2+4yz)dS=\int_0^2\int_0^{6-3z}[3z^2+4z\frac{1}{2}(6-x-3z)]\frac{\sqrt{14}}{2}dxdz$$

$$=\frac{\sqrt{14}}{2}\int_0^2[3z^2x-z(6-x-3z)^2]\Big|_0^{6-3z}dz$$

$$=\frac{\sqrt{14}}{2}\int_0^2(36z-18z^2)dz=\frac{\sqrt{14}}{2}(18z^2-6z^3)\Big|_0^2=12\sqrt{14}$$

3) Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ Assume C is oriented counterclockwise as

viewed from above. $\vec{F} = y^3\vec{i} - x^3\vec{j} + z^3\vec{k}$; C is the trace of the cylinder $x^2 + y^2 = 1$ in the plane $x + y + z = 1$ (Problem 9, page 534).

ANS curl $\vec{F} = (-3x^2 - 3y^2)\vec{k}$ A unit vector normal to the plane is $\vec{n} = (\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$ From

$z = 1 - x - y$, we have $z_x = z_y = -1$ and $dS = \sqrt{3}dA$ Then, using polar coordinates,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (curl\ \vec{F}) \cdot \vec{n}dS = \iint_R (-\sqrt{3}x^2 - \sqrt{3}y^2)\sqrt{3}dA$$

$$= 3\iint_R (-x^2 - y^2)dA = 3\int_0^{2\pi}\int_0^1 (-r^2)rdrd\theta$$

$$= 3\int_0^{2\pi} -\frac{1}{4}r^4\Big|_0^1 d\theta = 3\int_0^{2\pi} -\frac{1}{4}d\theta = -\frac{3\pi}{2}$$

4) Use the divergence theorem to find the outward flux $\iint_S (\vec{F} \cdot \vec{n})dS$ of the given vector filed $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$; D the region bounded by the sphere $x^2 + y^2 + z^2 = a^2$ (Problem 3, page 550).

ANS div $\vec{F} = 3x^2 + 3y^2 + 3z^2$ Using spherical coordinates,

$$\iint_S \vec{F}.\vec{n}dS = \iiint_D 3(x^2 + y^2 + z^2)dV = \int_0^{2\pi}\int_0^{\pi}\int_0^a 3\rho^2\rho^2 \sin\phi d\rho d\phi d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{3}{5}\rho^5 \sin\phi\Big|_0^a d\phi d\theta = \frac{3a^5}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi d\phi d\theta$$

$$= \frac{3a^5}{5}\int_0^{2\pi} -\cos\phi\Big|_0^{\pi} d\theta = \frac{6a^5}{5}\int_0^{2\pi} d\theta = \frac{12\pi a^5}{5}$$