HOMEWORK #6s (12.1 ~12.2)

Due on April 19

1) Show the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set. (Problem 12, page 656)

$$\left\{1, \cos\frac{n\pi}{p}x, \sin\frac{m\pi}{p}x\right\}, \qquad n = 1,2,3,\ldots, \qquad m = 1,2,3,\ldots;[-p, \quad p]$$

ANS For $m \neq n$

$$\int_{-p}^{p} \cos\frac{n\pi}{p}x \cos\frac{m\pi}{p}xdx = 2\int_{0}^{p} \cos\frac{n\pi}{p}x \cos\frac{m\pi}{p}xdx = 0$$

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x \sin\frac{m\pi}{p}xdx = 2\int_{0}^{p} \sin\frac{n\pi}{p}x \sin\frac{m\pi}{p}xdx = 0$$

Also

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x \cos\frac{m\pi}{p}xdx = \frac{1}{2}\int_{-p}^{p} (\sin\frac{(n-m)\pi}{p}x + \sin\frac{(n+m)\pi}{p}x)dx = 0,$$

$$\int_{-p}^{p} 1\cdot\cos\frac{n\pi}{p}xdx = \frac{p}{n\pi}\sin\frac{n\pi}{p}x\Big|_{-p}^{p} = 0,$$

$$\int_{-p}^{p} 1\cdot\sin\frac{n\pi}{p}xdx = -\frac{p}{n\pi}\cos\frac{n\pi}{p}x\Big|_{-p}^{p} = 0,$$

and

$$\int_{-p}^{p} \sin\frac{n\pi}{p}x \cos\frac{n\pi}{p}xdx = \int_{-p}^{p} \frac{1}{2}\sin\frac{2n\pi}{p}xdx = -\frac{p}{4n\pi}\cos\frac{2n\pi}{p}x\Big|_{-p}^{p} = 0$$

For $m = n$

$$\int_{-p}^{p} \cos^2\frac{n\pi}{p}xdx = \int_{-p}^{p} (\frac{1}{2} + \frac{1}{2}\cos\frac{2n\pi}{p}x)dx = p,$$

$$\int_{-p}^{p} \sin^2\frac{n\pi}{p}xdx = \int_{-p}^{p} (\frac{1}{2} - \frac{1}{2}\cos\frac{2n\pi}{p}x)dx = p,$$

and

$$\int_{-p}^{p} 1^2 dx = 2p$$

so that

$$\|1\| = \sqrt{2p}, \quad \left\|\cos\frac{n\pi}{p}x\right\| = \sqrt{p}, \text{ and } \left\|\sin\frac{n\pi}{p}x\right\| = \sqrt{p}$$

2) Verify by direct integration that the functions are orthogonal with respect to the indicated

weight function on the given interval (Problem 13, page 656)

$$H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2; \quad w(x) = e^{-x^2}, (-\infty, \infty)$$

ANS Since

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot 2x dx = -e^{-x^2} \Big|_{-\infty}^{0} - e^{-x^2} \Big|_{0}^{\infty} = 0,$$

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot 1 \cdot (4x^2 - 2) dx = 2\int_{-\infty}^{\infty} x(2xe^{-x^2}) dx - 2\int_{-\infty}^{\infty} e^{-x^2} dx = 2(-xe^{-x^2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2} dx) - 2\int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= 2(-xe^{-x^2} \Big|_{-\infty}^{0} - xe^{-x^2} \Big|_{0}^{\infty}) = 0,$$

and

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot 2x \cdot (4x^2 - 2) dx = 4\int_{-\infty}^{\infty} x^2 (2xe^{-x^2}) dx - 4\int_{-\infty}^{\infty} xe^{-x^2} dx$$

$$= 4(-x^2 e^{-x^2} \Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} xe^{-x^2} dx) - 4\int_{-\infty}^{\infty} xe^{-x^2} dx$$

$$= 4(-x^2 e^{-x^2} \Big|_{-\infty}^{0} - x^2 e^{-x^2} \Big|_{0}^{\infty}) + 2\int_{-\infty}^{\infty} 2xe^{-x^2} dx = 0,$$

the functions are orthogonal

3) Find the Fourier series of f on the given interval (Problem 7, page 661)

$$f(x) = x + \pi, \quad -\pi < x < \pi$$

ANS $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \pi) dx = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + \pi) \cos nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

4) Use the result of 3) to show (Problem 19, page 661)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

ANS The function in Problem 3) is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f(\frac{\pi}{2}) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\frac{n\pi}{2} = \pi + 2(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$