HOMEWORK #7s(12.3 ~12.4)

Due on May 3

1) Expand the given function in an appropriate cosine or sine series. (Problem 23, page 668)

$$f(x) = |\sin x|, \qquad -\pi < x < \pi$$

ANS Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x dx = \frac{2}{\pi}(-\cos x)|_0^{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \sin x \cos nx dx = \frac{1}{\pi/2}\int_0^{\pi/2} [\sin(1+n)x + \sin(1-n)x] dx$$

$$= \frac{2}{\pi(1-n^2)}[1+(-1)^n] \quad \text{for} \quad n = 2, 3, 4, \cdots$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx dx = \frac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] dx$$

$$= -\frac{1}{\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= -2\frac{(-1)^n+1}{\pi(n^2-1)}$$

$$a_1 = \frac{1}{\pi/2}\int_0^{\pi/2} \sin 2x dx = 0$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x dx = 0$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2[1+(-1)^n]}{\pi(1-n^2)} \cos nx$$

2) Find the half-range cosine and sine expansion of the given function (Problem 31, page 668)

$$f(x)=\begin{cases} x, & 0<x<1 \\ 1, & 1\le x<2 \end{cases}$$

ANS $a_0=\int_0^1 x dx+\int_1^2 1dx=\frac{3}{2}$

$$a_n=\int_0^1 x\cos\frac{n\pi}{2}xdx=\frac{4}{n^2\pi^2}(\cos\frac{n\pi}{2}-1)$$

$$b_n=\int_0^1 x\sin\frac{n\pi}{2}xdx+\int_1^2 1\cdot\sin\frac{n\pi}{2}xdx=\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2}{n\pi}(-1)^{n+1}$$

$$f(x)=\frac{3}{4}+\sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos\frac{n\pi}{2}-1)\cos\frac{n\pi}{2}x$$

$$f(x)=\sum_{n=1}^{\infty}[\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{2}{n\pi}(-1)^{n+1}]\sin\frac{n\pi}{2}x$$

3) Expand the given function in a Fourier series (Problem 37, page 668)

$$f(x)=x+1,\quad 0<x<1$$

ANS $a_0=2\int_0^1 (x+1)dx=3$

$$a_n=2\int_0^1 (x+1)\cos 2n\pi xdx=0$$

$$b_n=2\int_0^1 (x+1)\sin 2n\pi xdx=-\frac{1}{n\pi}$$

$$f(x)=\frac{3}{2}-\sum_{n=1}^{\infty}\frac{1}{n\pi}\sin 2n\pi x$$

4) Proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m=1,\ \ k=10$, and the driving force $f(t)$ is as given. Assume that when f(t) is extended to the negative *t*-axis in a periodic manner, the resulting function is odd. (Problem 39, page 668)

$$f(t)=\begin{cases} 5, & 0<t<\pi \\ -5, & \pi<t<2\pi \end{cases};\quad f(t+2\pi)=f(t)$$

ANS We have

$$b_n=\frac{2}{\pi}\int_0^{\pi} 5\sin ntdt=\frac{10}{n\pi}[1-(-1)^n]$$

so that

$$f(t)=\sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt$$

Substituting the assumption $x_p(t)=\sum_{n=1}^{\infty} B_n \sin nt$ into the differential equation then

gives $x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10-n^2)\sin nt = \sum_{n=1}^{\infty} \frac{10[1-(-1)^n]}{n\pi}\sin nt$

and so $B_n = \frac{10[1-(-1)^n]}{n\pi(10-n^2)}$ Thus

$$x_p(t)=\frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt$$

5) Find the complex Fourier series of f on the given interval. (Problem 1, page 672)

$$f(x)=\begin{cases}-1, & -2<x<0\\ 1, & 0<x<2\end{cases}$$

ANS we make use of the following identities due to Euler's formula:

$$e^{in\pi}=e^{-in\pi}=(-1)^n,\quad e^{-2in\pi}=1,\quad e^{-in\pi/2}=(-i)^n$$

Identifying $p=2$ we have

$$c_n=\frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}dx=\frac{1}{4}[\int_{-2}^{0}(-1)e^{-in\pi x/2}dx+\int_{0}^{2}e^{-in\pi x/2}dx]$$

$$=\frac{i}{2n\pi}[-1+e^{in\pi}+e^{-in\pi}-1]=\frac{i}{2n\pi}[-1+(-1)^n+(-1)^n-1]=\frac{1-(-1)^n}{n\pi i}$$

and

$$c_0=\frac{1}{4}\int_{-2}^{2} f(x)dx=0$$

Thus

$$f(x)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1-(-1)^n}{in\pi}e^{in\pi x/2}$$