HOMEWORK #9s (Fourier Transform)

Due on May 24

1) Find the Fourier transform of $f(x) = 5[H(x-1) - H(x-3)]$

ANS $f(x) = 5[H(x-1) - H(x-3)] \to f(t) = \begin{cases} 5 & 1 \le t \le 3 \\ 0 & t < 1, t > 3 \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} 5[H(t-3) - H(t-1)]e^{-i\omega t}dt$$

$$= \int_{1}^{3} 5e^{-i\omega t}dt = \frac{5e^{-i\omega t}}{-i\omega}\Big|_1^3$$

$$= -\frac{5e^{-3i\omega}}{i\omega} + \frac{5e^{-i\omega}}{i\omega} = -5\frac{e^{-3i\omega} - e^{-1i\omega}}{i\omega}$$

$$= -5e^{-2i\omega}\frac{e^{-i\omega} - e^{i\omega}}{i\omega} = 5e^{-2i\omega}\frac{2i\sin(\omega)}{i\omega} = 10e^{-2i\omega}\frac{\sin(\omega)}{\omega}$$

2) Find the Fourier transform of $f(x) = xe^{-a|x|} \qquad a > 0$

ANS $F\{f(x)\} = \int_{-\infty}^{0} xe^{ax}e^{-iwx}dx + \int_{0}^{\infty} xe^{-ax}e^{-iwx}dx = \int_{-\infty}^{0} xe^{(a-iw)x}dx + \int_{0}^{\infty} xe^{(-a-iw)x}dx$

$$= [\frac{x}{a-iw}e^{(a-iw)x} - \frac{1}{(a-iw)^2}e^{(a-iw)x}\Big|_{-\infty}^{0} + \frac{x}{-a-iw}e^{(-a-iw)x} - \frac{1}{(-a-iw)^2}e^{(-a-iw)x}\Big|_{0}^{\infty}]$$

$$= -\frac{1}{(a-iw)^2} + \frac{1}{(-a-iw)^2} = \frac{-4iaw}{(a^2-w^2)^2 + 4a^2w^2} = \frac{-4iaw}{(a^2+w^2)^2}$$

3) Find the Fourier transform of $f(t) = 3e^{-4|t|}\cos(2t)$

ANS $f(t) = 3e^{-4|t|}\cos(2t) = g(t)\cos(2t) \to g(t) = 3e^{-4|t|}$

$$F\{g(t)\} = \int_{-\infty}^{0} 3e^{4t}e^{-iwt}dt + \int_{0}^{\infty} 3e^{-4t}e^{-iwt}dt$$

$$= 3[\int_{-\infty}^{0} e^{(4-iw)t}dt + \int_{0}^{\infty} e^{(-4-iw)t}dt]$$

$$= 3[\frac{1}{(4-iw)} e^{(4-iw)t} \Big|_{-\infty}^{0} + \frac{1}{(-4-iw)} e^{(-4-iw)t} \Big|_{0}^{\infty}]$$

$$= 3[\frac{1}{4-iw} + \frac{1}{4+iw}]$$

$$= \frac{24}{16+w^2}$$

$$F\{g(t)\cos(2t)\} = \frac{1}{2}[F(w-2)+F(w+2)] = \frac{1}{2}[\frac{24}{16+(w-2)^2} + \frac{24}{16+(w+2)^2}]$$

$$= \frac{12}{16+(w-2)^2} + \frac{12}{16+(w+2)^2}$$

4) Solve $\frac{d^2y(t)}{dt^2} + 9y(t) = \cos(\omega_0 t)$ using fourier transform and discuss your result if $\omega_0 = 3$

ANS

First, if $F\{f(t)\} = F(\omega)$ and ω_0 is a real number, then

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$$

Also $F\{1\} = 2\pi\delta(\omega)$

$$\Rightarrow F\{\cos(\omega_0 t)\} = \frac{1}{2}[2\pi\delta(\omega+\omega_0) + 2\pi\delta(\omega-\omega_0)]$$

$$= \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

Hence

$$\frac{d^2y(t)}{dt^2} + 9y(t) = \cos(\omega_0 t) \Rightarrow F\left\{\frac{d^2y(t)}{dt^2} + 9y(t)\right\} = F\{\cos(\omega_0 t)\}$$

$$(\mathrm{i}\omega)^2 F\{y(t)\} + 9F\{y(t)\} = F\{\cos(\omega_0 t)\}$$

$$\Rightarrow F\{y(t)\} = \frac{F\{\cos(\omega_0 t)\}}{(\mathrm{i}\omega)^2 + 9}$$

$$= \frac{\pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)}{9-\omega^2}$$

$$
\begin{aligned}
y(t) &= F^{-1}\{F\{y(t)\}\} \\
&= F^{-1}\left\{\frac{\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)}{9-\omega^2}\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)}{9-\omega^2}e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\left\{\left[\frac{\pi}{9-\omega^2}e^{i\omega t}\right]\Big|_{\omega=\omega_0}+\left[\frac{\pi}{9-\omega^2}e^{i\omega t}\right]\Big|_{\omega=\omega_0}\right\} \\
&= \frac{1}{2\pi}\left\{\frac{\pi}{9-\omega_0^{\ 2}}e^{-i\omega_0 t}+\frac{\pi}{9-\omega_0^{\ 2}}e^{i\omega_0 t}\right\} \\
&= \frac{1}{2}\frac{e^{-i\omega_0 t}+e^{i\omega_0 t}}{9-\omega_0^{\ 2}} = \frac{\cos\omega_0 t}{9-\omega_0^{\ 2}}
\end{aligned}
$$

If $\omega_0 = 3$

$$y(t) = \frac{\cos\omega_0 t}{9-\omega_0^{\ 2}}\Big|_{\omega_0=3} \text{-->} \infty$$ ➔Resonant