Let $f(t)$ be a function defined for $t \geq 0$. Then the integral

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

is said to be the Laplace transform of a function $f(t)$, provided the integral converges.

If $F(s)$ represents the Laplace transform of a function $f(t)$, that is,

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$$

We then say $f(t)$ is the inverse Laplace transform of $F(s)$ and write

$$f(t) = L^{-1}\{F(s)\}$$

1)

$$\text{L}[t](s)=F(s)=\int_0^{\infty} e^{-st}tdt = \lim_{k\leftarrow\infty}\left[-\frac{te^{-st}}{s}\Big|_0^k - \int_0^k -\frac{e^{-st}}{s}d\right]$$

$$= \lim_{k\leftarrow\infty}\left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^k = \lim_{k\to\infty}\left[-\frac{ke^{-sk}}{s} - \frac{e^{-sk}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

$$\text{L}[t^n](s)=F(s)=\int_0^{\infty} e^{-st}t^n dt = \lim_{k\leftarrow\infty}\left[-\frac{t^n e^{-st}}{s}\Big|_0^k + \frac{n}{s}\int_0^k -e^{-st}t^{n-1}d\right]$$

$$= \frac{n}{s}\int_0^{\infty} e^{-st}t^{n-1}dt$$

➔ $\text{L}[t^n](s)=F(s)= \frac{n}{s}\int_0^{\infty} e^{-st}t^{n-1}dt = \frac{n!}{s^{n+1}}$

2)

$\text{L}[f'](s) = sF(s) - f(0)$

Set $f = t$ ➔ $f' = 1, \quad f(0) = 0$

➔ $\text{L}[1](s) = \frac{1}{s} = sF(s) - f(0) = sF(s)$

➔ $\text{L}[t](s) = F(s) = \frac{1}{s^2}$

$$\mathcal{L}[f''](s) = s^2F(s) - sf(0) - f'(0)$$

Set $f = t^2$ ➔ $f' = 2t$, $f'' = 2$, $f(0) = 0$

➔ $\mathcal{L}[2](s) = \frac{2}{s} = s^2F(s) - sf(0) - f(0)' = s^2F(s)$

➔ $\mathcal{L}[t^2](s) = F(s) = \frac{2}{s^3}$

3)

Set $f = te^{at}$ ➔ $f' = e^{at} + ate^{at}$, $f(0) = 0$

$\mathcal{L}[f'](s) = sF(s) - f(0)$

➔ $\mathcal{L}[e^{at} + ate^{at}](s) = sF(s) - f(0)$

➔ $\mathcal{L}[e^{at} + ate^{at}](s) = \mathcal{L}[e^{at}](s) + \mathcal{L}[ate^{at}](s) = s\mathcal{L}[te^{at}](s)$

➔ $\mathcal{L}[e^{at}](s) = (s-a)\,\mathcal{L}[te^{at}](s)$

➔ $\mathcal{L}[te^{at}](s) = \frac{1}{s-a}\,\mathcal{L}[e^{at}](s) = \frac{1}{(s-a)^2}$ $\quad (\mathcal{L}[e^{at}](s) = F(s) = \frac{1}{s-a})$

4)

$$\mathcal{L}[e^{at}](s) = F(s) = \frac{1}{s-a}$$

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2}$$

➔ $\mathcal{L}[\sinh(at)] = \frac{\frac{1}{s-a} - \frac{1}{s+a}}{2} = \frac{a}{s^2 - a^2}$

➔ $\mathcal{L}[\cosh(at)] = \frac{\frac{1}{s-a} + \frac{1}{s+a}}{2} = \frac{s}{s^2 - a^2}$

5)

➔ $\mathcal{L}[e^{iat}](s) = F(s) = \frac{1}{s-ia}$

$e^{iat} = \cos(at) + i\sin(at)$
$e^{-iat} = \cos(at) - i\sin(at)$ ➔ $\cos(at) = \frac{e^{iat} + e^{-iat}}{2}$, $\sin(at) = \frac{e^{iat} - e^{-iat}}{2i}$

➔ $\mathcal{L}[\sin(at)](s) = F(s) = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right] = \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$

➔ $\mathcal{L}[\cos(at)](s) = F(s) = \frac{1}{2}\left[\frac{1}{s-ia} + \frac{1}{s+ia}\right] = \frac{1}{2}\frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2}$

6)

Set $f = te^{iwt}$ ➔ $f' = e^{iwt} + iwte^{iwt}$, $f(0) = 0$

$\mathcal{L}[f'](s) = sF(s) - f(0)$

➔ $\mathcal{L}[e^{iwt} + iwte^{iwt}](s) = sF(s) - f(0)$

➔ $\mathcal{L}[e^{iwt} + iwte^{iwt}](s) = \mathcal{L}[e^{iwt}](s) + \mathcal{L}[iwte^{iwt}](s) = s\mathcal{L}[te^{iwt}](s)$

➔ $\mathcal{L}[e^{iwt} + iwte^{iwt}](s) = \mathcal{L}[e^{iwt}](s) + \mathcal{L}[iwte^{iwt}](s) = s\mathcal{L}[te^{iwt}](s)$

➔ $\mathcal{L}[te^{iwt}](s) = \frac{1}{s - iw}\mathcal{L}[e^{iwt}](s) = \frac{1}{(s - iw)^2}$

➔ $\mathcal{L}[te^{iwt}](s) = \mathcal{L}[t\cos(wt) + it\sin(wt)](s) = \frac{1}{(s - iw)^2}$

➔ $\mathcal{L}[t\cos(wt)](s) + i\mathcal{L}[t\sin(wt)](s) = \frac{1}{(s - iw)^2} = \frac{1}{s^2 - 2iws - w^2} = \frac{s^2 - w^2 + 2iws}{(s^2 - w^2)^2 - (2iws)^2}$

➔ $\mathcal{L}[t\cos(wt)](s) + i\mathcal{L}[t\sin(wt)](s) = \frac{s^2 - w^2 + 2iws}{(s^2 - w^2)^2 - (2iws)^2} = \frac{s^2 - w^2}{(s^2 + w^2)^2} + i\frac{2ws}{(s^2 + w^2)^2}$

➔ $\mathcal{L}[t\cos(wt)](s) = \frac{s^2 - w^2}{(s^2 + w^2)^2}$

➔ $\mathcal{L}[t\sin(wt)](s) = \frac{2ws}{(s^2 + w^2)^2}$

7)

$\mathcal{L}[t^{-\frac{1}{2}}](s) = F(s) = \int_0^\infty e^{-st}t^{-\frac{1}{2}}dt$ (令 u=st)

$$= \int_0^\infty e^{-u}\left(\sqrt{\frac{s}{u}}\right)\left(\frac{du}{s}\right) = \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{-\frac{1}{2}}du$$

$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-u}u^{-\frac{1}{2}}du \quad (u=T^2)$$

$$= \frac{1}{\sqrt{s}}\int_0^\infty e^{-T^2}\left(\frac{1}{T}\right)2TdT =$$

$$\frac{1}{\sqrt{s}}\left(\int_0^\infty 2e^{-T^2}dT\right) = \frac{1}{\sqrt{s}}\left(2\frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}}$$

$(\int_0^\infty e^{-x^2}dx = ?)$

$$==> I = \int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-y^2}dy$$

➔ $I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx = \iint_S e^{-(x^2+y^2)} dA$ s={(x,y)|x $\geq 0, y \geq 0$ }

➔ $I^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-(r)^2} drd\theta = \frac{\pi}{4}$

➔ $I = \frac{\sqrt{\pi}}{2}$