# 海洋大學河海工程學系九十四學年度 第二學期 ■期中 □小考 考試命題紙

| 考 試 科 目 | 開課系級 | 考試日期 | 印製份數 | 答案紙 | 命題教師 | 備 註 |
|---|---|---|---|---|---|---|
| 工程數學二 | 二 A, B | 5 月 12 日 | 110 | ■ 需<br>□不需 | 陳桂鴻<br>呂學育 | 第二次大考 |

1. $y''(t)+\omega^2 y(t)=F(t)$ where $F(t)=\begin{cases}1, & t\in(0,\pi)\\ 0, & t\in(\pi,2\pi)\end{cases}$ and $F(t)=F(t+2\pi)$. (20%)

(a) Find $y_p(t)$ by using the complex Fourier expansion. (10%)

(b) Plot the frequency spectrum of $y_p(t)$. (5%)

(c) Choice the right answer and explain why when cause the phenomenon of Resonance as (5%)

(1) $\omega$ is odd numbers. (2) $\omega$ is even numbers.

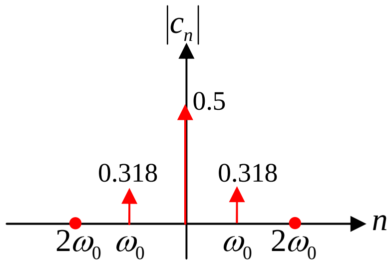
(3) $\omega$ is integer numbers. (4) The resonance will not occur.

**ANS** (a) $F(t)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$ ; $c_n=\frac{1}{2\pi}\int_0^{\pi} e^{-inx}dx=\frac{-1}{2in\pi}\left[(-1)^n-1\right]$

$$y_p(t)=\sum_{n=-\infty}^{\infty}\frac{-1}{2in\pi(-n^2+\omega^2)}\left[(-1)^n-1\right]e^{inx}$$

(b) $c_0=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}$ ;

| $n$ | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $\lvert c_n\rvert$ | 0 | 0.318 | 0.5 | 0.318 | 0 |




(c) (3) $\omega$ is integer numbers.

2. Suppose a uniform beam of length $L$ is simply supported at x=0 and at x=$L$. If the load per unit length is given by $r(x)=\begin{cases}0, & 0<x<L\\ w_0(x-L), & L<x<2L\\ 0, & 2L<x<3L\end{cases}$, $0<x<3L$, $r(x+3L)=r(x)$, and then the differential equation for the deflection $y(x)$ is $EI\frac{d^4y}{dx^4}=r(x)$, where $E$, $I$, and $w_0$ are constants. (40%)

(a) Find the homogenous solution $y_h$. (5%)

(b) Expand $r(x)$ in a half-range cosine series. (7%)

(c) Find a particular solution $y_p(x)$ by using the Fourier series expansion.(10%)

(d) Expand $r(x)$ in a complex Fourier series and plot frequency spectrum of $r(x)$. (8%)

(e) Find a particular solution $y_p(x)$ by using the complex Fourier series expansion. (10%)

**ANS** (a) $y_h=c_1+c_2x+c_3x^2+c_4x^3$

(b) $r(x)=a_0+\sum_{n=0}^{\infty}a_n\cos\frac{n\pi}{3L}x$; $a_0=\frac{1}{6L}\int_{-3L}^{3L}r(x)dx=\frac{-\omega_0 L}{3}$,

$$a_n=\frac{1}{3L}\int_{-3L}^{3L}r(x)\cos\frac{n\pi}{3L}xdx=\frac{2w_0L}{n\pi}\sin\frac{2n\pi}{3}+\frac{6w_0L}{(n\pi)^2}(\cos\frac{2n\pi}{3}-\cos\frac{n\pi}{3})$$

(c) $r(x)=a_0+\sum_{n=0}^{\infty}a_n\cos\frac{2n\pi}{3L}x+b_n\sin\frac{2n\pi}{3L}x$

$$a_0 = \frac{1}{3L}\int_0^{3L} r(x)dx = \frac{w_0 L}{6}\;;\quad a_n = \frac{2}{3L}\int_0^{3L} r(x)\cos\frac{2n\pi}{3L}dx = \frac{w_0 L}{n\pi}(\sin\frac{4n\pi}{3}) + \frac{3w_0 L}{(n\pi)^2}(\cos\frac{4n\pi}{3} - \cos\frac{2n\pi}{3})\;;$$

$$b_n = \frac{2}{3L}\int_0^{3L} r(x)\sin\frac{2n\pi}{3L}dx = \frac{-8w_0 L}{n\pi}(\cos\frac{4n\pi}{3}) + \frac{w_0 L}{(n\pi)^2}(\sin\frac{4n\pi}{3} - \sin\frac{2n\pi}{3})$$

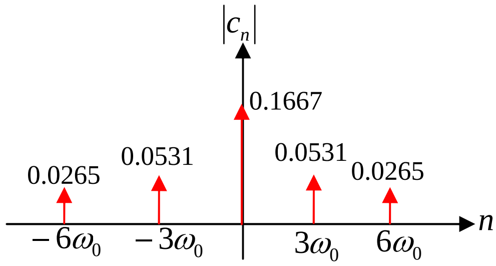
Let $y_p(x) = A_0 x^4 + \sum_{n=0}^{\infty} A_n \cos\frac{2n\pi}{3L}x + B_n \sin\frac{2n\pi}{3L}x$

$$A_0 = \frac{w_0 L}{144EI}\,,\quad A_n = (\frac{3L}{2n\pi})^2 \frac{1}{EI}a_n\,,\quad B_n = (\frac{3L}{2n\pi})^4 \frac{1}{EI}b_n$$

(d) $r(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2n\pi}{3L}x}\;;\quad c_n = \frac{1}{3L}\int_{-3L/2}^{3L/3} r(x)e^{-i\frac{2n\pi}{3L}x}dx = \frac{-w_0 L}{2n\pi} i e^{-i\frac{4n\pi}{3}} + \frac{3w_0 L}{(2n\pi)^2}(e^{-i\frac{4n\pi}{3}} - e^{-i\frac{2n\pi}{3}})$

$$c_0 = \frac{1}{3L}\int_L^{2L} w_0(x-L)dx = \frac{w_0 L}{6}$$

| $n$ | -6 | -3 | 0 | 3 | 6 |
|---|---|---|---|---|---|
| $\lvert c_n \rvert$ | 0.0265 | 0.0531 | 0.1667 | 0.0531 | 0.0265 |




(e)Let $y_p(x) = \sum_{n=-\infty}^{\infty} d_n e^{i\frac{2n\pi}{3L}x}\;;\quad d_n = (\frac{3L}{2n\pi})^4 \frac{1}{EI}c_n$

3. Consider $f(x) = x + \pi\,,\quad -\pi < x < \pi$

(1) determine whether the function $f$ is even, odd, or neither (3 scores)

**ANS** neither

(2) find the Fourier series of $f$ on the given interval (-π, π) (8 scores)

**ANS** $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)dx = 2\pi$

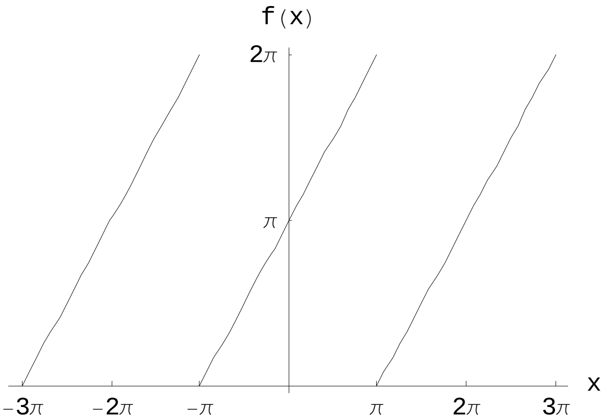
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\cos nx dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+\pi)\sin nx dx = \frac{-2}{n}(-1)^n$$

$$f(x) = \pi + \sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n \sin nx$$

(3) give the values that the series will converge at $x$ = -π, 0, π/2, π (4 scores)

**ANS**




$$\frac{f(-\pi^+)+f(-\pi^-)}{2}=\frac{0+2\pi}{2}=\pi$$

$$\frac{f(0^+)+f(0^-)}{2}=\frac{\pi+\pi}{2}=\pi$$

$$\frac{f(\frac{\pi}{2}^+)+f(\frac{\pi}{2}^-)}{2}=\frac{\frac{3\pi}{2}+\frac{3\pi}{2}}{2}=\frac{3\pi}{2}$$

$$\frac{f(\pi^+)+f(\pi^-)}{2}=\frac{0+2\pi}{2}=\pi$$

(4) use the result of (2) to show $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ (5 scores)

**ANS** Let $x=\frac{\pi}{2}\rightarrow\frac{\pi}{2}+\pi=\pi+\sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx\rightarrow\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx$

$$\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{-2}{n}(-1)^n\sin nx=2\cdot1+\frac{2}{3}\cdot(-1)+\frac{2}{5}\cdot1+\frac{2}{7}\cdot(-1)+\cdots$$

$$\therefore\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

4. Expand $f(\mathrm{x})=\begin{cases} x & for \quad 0\le x\le L/2 \\ L-x & for \quad L/2<x\le L\end{cases}$

(1) in a sine series AND give the value that the series will converge at $x=L$ (10 scores)

**ANS** $a_0=\frac{2}{L}\int_0^{\frac{L}{2}}xdx+\frac{2}{L}\int_{\frac{L}{2}}^{L}(L-x)dx=L/2$

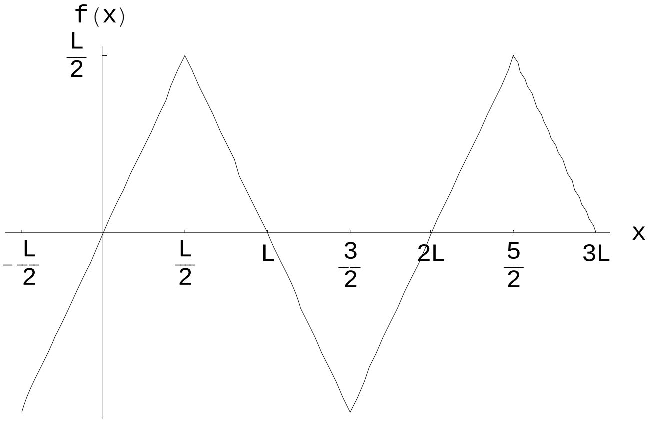
$$a_n=\frac{2}{L}\int_0^{\frac{L}{2}}x\cdot\cos\frac{n\pi x}{L}dx+\frac{2}{L}\int_{\frac{L}{2}}^{L}(L-x)\cdot\cos\frac{n\pi x}{L}dx=\frac{8L\cos\frac{n\pi}{2}\sin^2\frac{n\pi}{4}}{n^2\pi^2}$$

$$=\frac{2L(-1+2\cos\frac{n\pi}{2}-(-1)^n)}{n^2\pi^2}$$

$$b_n = \frac{2}{L}\int_0^{\frac{L}{2}} x\cdot\sin\frac{n\pi x}{L}dx + \frac{2}{L}\int_{\frac{L}{2}}^{L}(L-x)\cdot\sin\frac{n\pi x}{L}dx = \frac{16L\cos\frac{n\pi}{4}\sin^3\frac{n\pi}{4}}{n^2\pi^2}$$

$$= \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

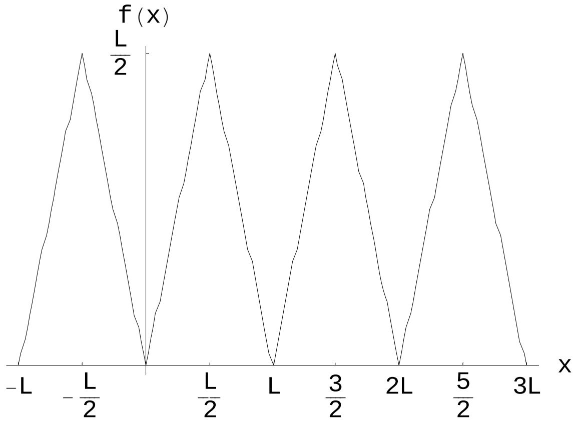
$$f(x) = \sum_{n=1}^{\infty}\frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}\cdot\sin\frac{n\pi x}{L}$$




$$\frac{f(L^+)+f(-L^-)}{2} = \frac{0+0}{2} = 0$$

(2) in a cosine series AND give the value that the series will converge at $x = L$ (10 scores)

**ANS** $f(x) = L/4 + \sum_{n=1}^{\infty}\frac{2L(-1+2\cos\frac{n\pi}{2}-(-1)^n)}{n^2\pi^2}\cdot\cos\frac{n\pi x}{L} = L/4$




$$\frac{f(L^+)+f(-L^-)}{2} = \frac{0+0}{2} = 0$$

5. $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$

(1) find the complex Fourier series of $f$ on the given interval (10 scores)

**ANS** we make use of the following identities due to Euler's formula:

$e^{in\pi} = e^{-in\pi} = (-1)^n$, $e^{-2in\pi} = 1$, $e^{-in\pi/2} = (-i)^n$

Identifying $p = 2$ we have

$$c_n = \frac{1}{4}\int_{-2}^{2} f(x)e^{-in\pi x/2}dx = \frac{1}{4}[\int_{-2}^{0}(-1)e^{-in\pi x/2}dx + \int_{0}^{2} e^{-in\pi x/2}dx]$$

$$= \frac{i}{2n\pi}[-1+e^{in\pi}+e^{-in\pi}-1] = \frac{i}{2n\pi}[-1+(-1)^n+(-1)^n-1] = \frac{1-(-1)^n}{n\pi i}$$

and

$$c_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = 0$$

Thus

$$f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1-(-1)^n}{in\pi}e^{in\pi x/2}$$

(2) find the frequency spectrum of the periodic wave that is the periodic extension of the function $f$ (10 scores)

**ANS** The fundamental period is $T=4$, so $w = 2\pi/4 = \pi/2$ and the values of $nw$ are 0, $\pm\pi/2$, $\pm\pi$, $\pm 3\pi/2, \cdots$. From Problem 1, $c_0 = 0$

and $|c_n| = (1-(-1)^n)/n\pi$. The table shows some values of $n$ with corresponding

values of $|c_n|$. The graph is a portion of the frequency spectrum.

| $n$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_n$ | 0.1273 | 0.0000 | 0.2122 | 0.0000 | 0.6366 | 0.0000 | 0.6366 | 0.0000 | 0.2122 | 0.0000 | 0.1273 |