Proof of some important Fourier Transform properties

1)Theorem Scaling

If a is a nonzero real number, then

$$F\{f(at)\}=\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

(Note that we denote $F\{f(t)\}=F(\omega)$)

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt$$

1) If $a>0$

Set $at=s$, ==> $adt=ds$

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt=\int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{a}=\int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{|a|}$$

2) If $a<0$

Set $-|a|t=s$, ==> $-|a|dt=ds$, Note that

$$\int_{-\infty}^{\infty} f(at)e^{-iwt}dt=-\int_{\infty}^{-\infty} f(s)e^{-iws/a}\frac{ds}{|a|}=\int_{-\infty}^{\infty} f(s)e^{-iws/a}\frac{ds}{|a|}$$

$$\Rightarrow\int_{-\infty}^{\infty} f(at)e^{-iwt}dt=\frac{1}{|a|}\int_{-\infty}^{\infty} f(s)e^{-i\frac{w}{a}s}ds=\frac{1}{|a|}\int_{-\infty}^{\infty} f(t)e^{-i\frac{w}{a}t}dt=\frac{1}{|a|}F\left(\frac{w}{a}\right)$$

2)Theorem Time Shifting

If t_0 is a real number, then

$$\begin{aligned}F[f(t-t_0)]&=\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega t}dt\\&=e^{-i\omega t_0}\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega(t-t_0)}dt\end{aligned}$$

Let $u=t-t_0$ ==> $du=dt$

$$\begin{aligned}F[f(t-t_0)]&=e^{-i\omega t_0}\int_{-\infty}^{\infty} f(t-t_0)e^{-i\omega(t-t_0)}dt\\&=e^{-i\omega t_0}\int_{-\infty}^{\infty} f(u)e^{-i\omega u}du\\&=e^{-i\omega t_0}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\\&=e^{-i\omega t_0}F[f(t)]\end{aligned}$$

3)Theorem Frequency Shifting

If ω_0 is any number, then

$$
\begin{aligned}
F\left\{e^{i\omega_0 t} f(t)\right\} &= \int_{-\infty}^{\infty} e^{i\omega_0 t} f(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{\infty} f(t) e^{-i(\omega-\omega_0)t} dt \\
&= F(\omega-\omega_0)
\end{aligned}
$$

Note that

$$
\begin{aligned}
F\{f(t)\} &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= F(\omega)
\end{aligned}
$$

4)Theorem *Time Reversal*

$$F\{f(\text{-}t)\} = F(\text{-}\omega)$$

$$F\{f(-t)\} = \int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt$$

Let $u = -t$ ==> $du = -dt$

$$
\begin{aligned}
F\{f(-t)\} &= \int_{-\infty}^{\infty} f(-t) e^{-i\omega t} dt \\
&= \int_{\infty}^{-\infty} f(u) e^{i\omega u} (-du) \\
&= \int_{-\infty}^{\infty} f(u) e^{-i(-\omega)u} du \\
&= \int_{-\infty}^{\infty} f(t) e^{-i(-\omega)t} dt = F(-\omega)
\end{aligned}
$$

➔

5)Theorem *Symmetry*

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad ➔$$

$$
\begin{aligned}
F\{F(\omega)\} &= \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} dt \\
&= 2\pi f(-w)
\end{aligned}
$$

By the formula for the inverse Fourier transform

$$f(t) = F^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$
\begin{aligned}
=> 2\pi f(t) &= \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \\
&= \int_{-\infty}^{\infty} F(s) e^{ist} ds
\end{aligned}
$$

$$
\begin{aligned}
=> 2\pi f(-\omega) &= \int_{-\infty}^{\infty} F(s) e^{-i\omega s} ds \qquad (\text{let} \quad t = \text{-}\omega) \\
&= \int_{-\infty}^{\infty} F(t) e^{-i\omega t} dt = F\{F(t)\}
\end{aligned}
$$

6)Theorem *Modulation*

First, we know $F\{f(t)\}=F(\omega)$ and $F\{1\}=2\pi\delta(\omega)$.

If ω_0 is a real number, then

1) $$\boxed{F\{f(t)\cos(\omega_0 t)\}=\frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]}$$

Pf: Note that $\cos(\omega_0 t)=\dfrac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}$

$$F\{f(t)\cos(\omega_0 t)\}=\int_{-\infty}^{\infty} f(t)\left[\frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2}\right]e^{-i\omega t}dt$$

$$\Rightarrow \quad =\int_{-\infty}^{\infty}\left(e^{i\omega_0 t}f(t)+e^{-i\omega_0 t}f(t)\right)e^{-i\omega t}dt$$

$$=\frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$$

If $f(t)=1$ ➔ $F\{\mathrm{f(t)}\}=F\{1\}=2\pi\delta(\omega)$

➔Find the fourier transform of $\cos(\omega_0 t)$

$$F\{\cos(\omega_0 t)\}=\frac{1}{2}[F(\omega+\omega_0)+F(\omega-\omega_0)]$$

$$=\frac{1}{2}[2\pi\delta(\omega+\omega_0)+2\pi\delta(\omega-\omega_0)]$$

$$=\pi\delta(\omega+\omega_0)+\pi\delta(\omega-\omega_0)$$

2) $$\boxed{F\{f(t)\sin(\omega_0 t)\}=\frac{1}{2}[F(\omega+\omega_0)-F(\omega-\omega_0)]}$$

Pf: Note that $\sin(\omega_0 t)=\dfrac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}$

$$F\{f(t)\sin(\omega_0 t)\}=\int_{-\infty}^{\infty} f(t)\left[\frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}\right]e^{-i\omega t}dt$$

$$\Rightarrow \quad =\frac{1}{2i}\int_{-\infty}^{\infty}\left(e^{i\omega_0 t}f(t)-e^{-i\omega_0 t}f(t)\right)e^{-i\omega t}dt$$

$$=\frac{1}{2i}[F(\omega-\omega_0)-F(\omega+\omega_0)]$$

$$=\frac{i}{2}[F(\omega+\omega_0)-F(\omega-\omega_0)]$$

If $f(t)=1$ ➔ $F\{\mathrm{f(t)}\}=F\{1\}=2\pi\delta(\omega)$

➔Find the fourier transform of $\sin(\omega_0 t)$

$$F\{\sin(\omega_0 t)\} = \frac{1}{2} i[F(\omega+\omega_0) - F(\omega-\omega_0)]$$
$$= \frac{1}{2} i[2\pi\delta(\omega+\omega_0) - 2\pi\delta(\omega-\omega_0)]$$
$$= i[\pi\delta(\omega+\omega_0) - \pi\delta(\omega-\omega_0)]$$

Example:

Find the fourier transform of $g(t) = 3e^{-4|t|}\cos(2t) = f(t)\cos(2t)$ ➔ $f(t) = 3e^{-4|t|}$

➔ $F\{f(t)\} = F(\omega) = 3\dfrac{8}{16+\omega^2} = \dfrac{24}{16+\omega^2}$

Also by the property of Theorem *Modulation*

$$F\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)]$$

➔ $$= \frac{1}{2}\left[\frac{24}{16+(\omega-\omega_0)^2} + \frac{24}{16+(\omega+\omega_0)^2}\right] \quad \text{with } \boxed{\omega_0 = 2}$$

$$= \frac{12}{16+(\omega-\omega_0)^2} + \frac{12}{16+(\omega+\omega_0)^2}$$

Important Example: Solve for y(t) using fourier transform in the following ODE

$$\frac{d^2y(t)}{dt^2} + 3y(t) = \cos(5t)$$

➔ $$F\left\{\frac{d^2y(t)}{dt^2} + 3y(t)\right\} = F\{\cos(5t)\}$$

$$F\left\{\frac{d^2y(t)}{dt^2}\right\} + F\{3y(t)\} = F\{\cos(5t)\}$$

$$(\mathrm{i}\omega)^2 F\{y(t)\} + 3F\{y(t)\} = F\{\cos(5t)\}$$
$$F\{y(t)\} = \frac{F\{\cos(5t)\}}{(\mathrm{i}\omega)^2 + 3}$$
$$= \frac{\pi\delta(\omega+5) + \pi\delta(\omega-5)}{3-\omega^2}$$

$$
\begin{aligned}
y(t) &= F^{-1}\left\{F\{y(t)\}\right\} \\
&= F^{-1}\left\{\frac{\pi\delta(\omega+5)+\pi\delta(\omega-5)}{3-\omega^2}\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi\delta(\omega+5)+\pi\delta(\omega-5)}{3-\omega^2}e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\left\{\left[\frac{\pi}{3-\omega^2}e^{i\omega t}\right]\Big|_{\omega=-5}+\left[\frac{\pi}{3-\omega^2}e^{i\omega t}\right]\Big|_{\omega=5}\right\} \\
&= \frac{1}{2\pi}\left\{\frac{\pi}{3-25}e^{-i5t}+\frac{\pi}{3-25}e^{i5t}\right\} = -\left\{\frac{e^{-i5t}}{44}+\frac{e^{i5t}}{44}\right\} = -\frac{\cos 5t}{22}
\end{aligned}
$$