Review exercise 2s

1) A real-valued function f is said to be **periodic** with period T if $f(x+T)=f(x)$. For example, 4π is a period of $\sin x$ since $\sin(x+4\pi)=\sin x$. The smallest value of T for which $f(x+T)=f(x)$ holds is called the **fundamental period** of f. For example, the fundamental period of $f(x)=\sin x$ is $T=2\pi$. What is the fundamental period of each of the following functions ? (Exercise 12.1, problem 21)

(a) $f(x)=\cos 2\pi x$ (b) $f(x)=\sin\frac{4}{L}x$

(c) $f(x)=\sin x+\sin 2x$ (d) $f(x)=\sin 2x+\cos 4x$

(e) $f(x)=\sin 3x+\cos 2x$ (f) $f(x)=A_0+\sum_{n=1}^{\infty}(A_n\cos\frac{n\pi}{p}x+B_n\sin\frac{n\pi}{p}x)$, A_n and B_n depend only on n

ANS (a) The fundamental period is $2\pi/2\pi=1$

(b) The fundamental period is $2\pi/(4/L)=\frac{1}{2}\pi L$

(c) The fundamental period of $\sin x+\sin 2x$ is 2π

(d) The fundamental period of $\sin 2x+\cos 4x$ is $2\pi/2=\pi$

(e) The fundamental period of $\sin 3x+\cos 2x$ is 2π

(f) The fundamental period of $f(x)$ is $2\pi/(n\pi/p)=2p/n$

2) In this problem, find the Fourier series of f on the given interval

$$f(x)=\begin{cases}0, & -2<x<-1\\ -2, & -1\le x<0\\ 1, & 0\le x<1\\ 0, & 1\le x<2\end{cases}$$ (Exercise 12.2, Problem 11)

ANS $a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}(\int_{-1}^{0}-2dx+\int_{0}^{1}1dx)=-\frac{1}{2}$

$$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{n\pi}{2}xdx=\frac{1}{2}(\int_{-1}^{0}[-2\cos\frac{n\pi}{2}x]dx+\int_{0}^{1}\cos\frac{n\pi}{2}xdx)=-\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n=\frac{1}{2}\int_{-2}^{2}f(x)sin\frac{n\pi}{2}xdx=\frac{1}{2}(\int_{-1}^{0}[-2\sin\frac{n\pi}{2}x]dx+\int_{0}^{1}\sin\frac{n\pi}{2}xdx)=\frac{3}{n\pi}(1-\cos\frac{n\pi}{2})$$

$$f(x)=-\frac{1}{4}+\sum_{n=1}^{\infty}[-\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x+\frac{3}{n\pi}(1-\cos\frac{n\pi}{2})\sin\frac{n\pi}{2}x]$$

3) In this problem, expand the given function in an appropriate cosine or sine series

$f(x)=|x|, \quad -\pi < x < \pi$ (Exercise 12.3, Problem 13)

ANS Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2}{n^2\pi}[(-1)^n - 1]$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos nx$$

4) In this problem, expand the given function in an appropriate cosine or sine series
$f(x) = x^2, \quad -1 < x < 1$ (Exercise 12.3, Problem 15)

ANS Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = 2\int_0^1 x^2 dx = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2 \cos n\pi x dx = 2\left(\frac{x^2}{n\pi}\sin n\pi x\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x dx\right) = \frac{4}{n^2\pi^2}(-1)^n$$

Thus

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos n\pi x$$

5) In this problem, find the half-range cosine and sine expansions of the given function
$f(x) = x^2 + x, \quad 0 < x < 1$ (Exercise 12.3, Problem 33)

ANS $a_0 = 2\int_0^1 (x^2 + x)dx = \frac{5}{3}$

$$a_n = 2\int_0^1 (x^2+x)\cos n\pi x dx = \frac{2(x^2+x)}{n\pi}\sin n\pi x\Big|_0^1 - \frac{2}{n\pi}\int_0^1 (2x+1)\sin n\pi x dx$$

$$= \frac{2}{n^2\pi^2}[3(-1)^n - 1]$$

$$b_n = 2\int_0^1 (x^2+x)\sin n\pi x dx = -\frac{2(x^2+x)}{n\pi}\cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (2x+1)\cos n\pi x dx$$

$$= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

$$f(x) = \frac{5}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[3(-1)^n - 1]\cos n\pi x$$

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{4}{n\pi}(-1)^{n+1}+\frac{4}{n^3\pi^3}[(-1)^n-1]\right)\sin n\pi x$$

6) In this problem, expand the given function in a Fourier series
$f(x)=x^2,\ \ 0<x<2\pi$ (Exercise 12.3, Problem 35)

ANS $a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{8}{3}\pi^2$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx dx=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx dx=-\frac{4\pi}{n}$$

$$f(x)=\frac{4}{3}\pi^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$$

7) Find the frequency spectrum of the periodic wave that is the periodic extension of the function f in Problem 1 (Exercise 12.4, problem 7)

In Problem 1, find the complex Fourier series of f on the given interval

$$f(x)=\begin{cases}-1, & -2<x<0\\ 1, & 0<x<2\end{cases}$$

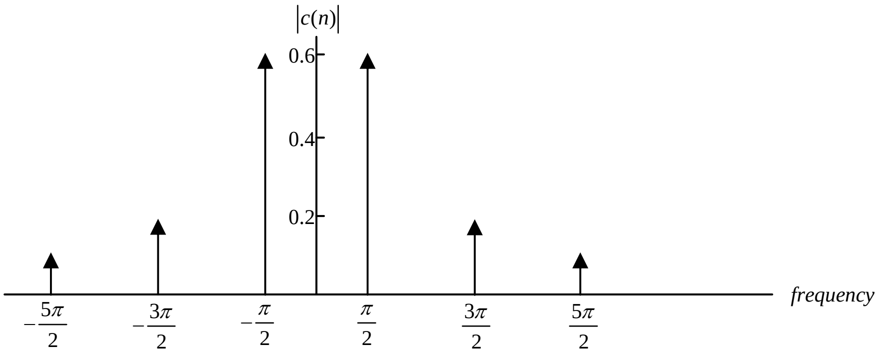
ANS The fundamental period is $T=4$, so $w=2\pi/4=\pi/2$ and the values of nw are 0, $\pm\pi/2$, $\pm\pi$, $\pm3\pi/2,\cdots$. From Problem 1, $c_0=0$

and $|c_n|=(1-(-1)^n)/n\pi$. The table shows some values of n with

corresponding

values of $|c_n|$. The graph is a portion of the frequency spectrum.

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
c_n	0.1273	0.0000	0.2122	0.0000	0.6366	0.0000	0.6366	0.0000	0.2122	0.0000	0.1273



8) Suppose the function $f(x)=x^2+1,\ \ 0<x<3$, is expanded in a Fourier series, a

cosine series, and a sine series. Give the value to which each series will converge at $x = 0$ (Chapter 12 Review Exercises, problem 7)

ANS The Fourier series will converge to 1, the cosine series to 1, and the sine series to 0 at $x = 0$. Respectively, this is because the rule $(x^2 + 1)$ defining $f(x)$ determines a continuous function on $(-3, 3)$, the even expansion of f to $(-3, 0)$ is continuous at 0, and the odd extension of f to $(-3, 0)$ approaches -1 as x approaches 0 from the left.