1)

The Heaviside function $H(t)$ is given by

$$H(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

Find the Fourier transform of $f(t)=H(t)e^{-at}$ with a a positive constant.

The Fourier transform is

$$\begin{aligned}F\{f(t)\}&=\int_{-\infty}^{\infty} H(t)e^{-at}e^{-i\omega t}dt\\ &=\int_{0}^{\infty} e^{-(a+i\omega)t}dt\\ &=-\frac{1}{a+i\omega}e^{-(a+i\omega)t}\Big|_0^{\infty}=\frac{1}{a+i\omega}\end{aligned}$$

2)

Determine the value of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{2i\omega}}{5+i\omega}e^{i\omega t}d\omega$

Note that this is to determine the inverse Fourier transform of $F(\omega)=\frac{e^{2i\omega}}{5+i\omega}$

➔ $F^{-1}\left\{\frac{1}{5+i\omega}\right\}=f(\mathrm{t})=\mathrm{H(t)e^{-5t}}$ ➔ $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{5+i\omega}e^{i\omega t}d\omega=f(\mathrm{t})=\mathrm{H(t)e^{-5t}}$

Applying the time time-shifting theorem on $f(t+2)$

➔ $F\{f(t+2)\}=\frac{e^{2i\omega}}{5+i\omega}$

Therefore $\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{2i\omega}}{5+i\omega}e^{i\omega t}d\omega=f(\mathrm{t}+2)=\mathrm{H(t+2)e^{-5(t+2)}}$

3)

Use the Fourier transform to find one solution $y(t)$ of

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\delta(t)$$

Apply the Fourier transform on the differential equation

$$(i\omega)^2F\{y(t)\}+3i\omega F\{y(t)\}+2F\{y(t)\}=1 \Rightarrow -\omega^2F\{y(t)\}+3i\omega F\{y(t)\}+2F\{y(t)\}=1$$

$$\Rightarrow F\{y(t)\}=\frac{1}{-\omega^2+3i\omega+2}$$

Therefore

$$y(t)=F^{-1}\left\{\frac{1}{-\omega^2+3i\omega+2}\right\}$$

Note that $-\omega^2+3i\omega+2=(1+i\omega)(2+i\omega)$

Then

$$\begin{aligned} y(t)&=F^{-1}\left\{\frac{1}{-\omega^2+3i\omega+2}\right\}\\ &=F^{-1}\left\{\frac{1}{(1+i\omega)(2+i\omega)}\right\}\\ &=F^{-1}\left\{\frac{1}{(1+i\omega)}-\frac{1}{(2+i\omega)}\right\}\\ &=H(t)e^{-t}-H(t)e^{-2t}\end{aligned}$$

Finally

$$y(t)=\begin{cases} e^{-t}-e^{-2t} & t\geq 0\\ 0 & t<0\end{cases}$$