Review exercise 4s

1) In this Problem , use the Laplace transform to solve the given system of differential equations $\frac{dx}{dt} = -x + y, \frac{dy}{dt} = 2x; \quad x(0) = 0, y(0) = 1$ (Problem 1, page 230)

ANS Taking the Laplace transform of the system gives

$$sL\{x\} = -L\{x\} + L\{y\},\ sL\{y\} - 1 = 2L\{x\}$$

so that

$$L\{x\} = \frac{1}{(s-1)(s+2)} = \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+2}$$

and

$$L\{y\} = \frac{1}{s} + \frac{2}{s(s-1)(s+2)} = \frac{2}{3}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s+2}$$

Then

$$x = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t} \quad \text{and} \quad y = \frac{2}{3}e^{t} + \frac{1}{3}e^{-2t}$$

2) In this Problem , use the Laplace transform to solve the given system of differential equations $2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1, \frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2; \quad x(0) = 0$, $y(0) = 0$ (Problem 5, page 230)

ANS Taking the Laplace transform of the system gives

$$(2s-2)L\{x\} + sL\{y\} = \frac{1}{s},\ (s-3)L\{x\} + (s-3)L\{y\} = \frac{2}{s}$$

so that

$$L\{x\} = \frac{-s-3}{s(s-2)(s-3)} = -\frac{1}{2}\frac{1}{s} + \frac{5}{2}\frac{1}{s-2} - \frac{2}{s-3}$$

and

$$L\{y\} = \frac{3s-1}{s(s-2)(s-3)} = -\frac{1}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{s-2} + \frac{8}{3}\frac{1}{s-3}$$

Then

$$x = -\frac{1}{2} + \frac{5}{2}e^{2t} - 2e^{3t} \quad \text{and} \quad y = -\frac{1}{6} - \frac{5}{2}e^{2t} + \frac{8}{3}e^{3t}$$

3) In this Problem , use the Laplace transform to solve the given system of differential equations $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = t^2, \quad \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = 4t; \quad x(0) = 8, x'(0) = 0,$

$y(0)=0,\ y'(0)=0$ (Problem 9, page 231)

ANS Adding the equations and then subtracting them gives

$$\frac{d^2x}{dt^2}=\frac{1}{2}t^2+2t\,,\quad \frac{d^2y}{dt^2}=\frac{1}{2}t^2-2t$$

Taking the Laplace transform of the system gives

$$L\{x\}=8\frac{1}{s}+\frac{1}{24}\frac{4!}{s^5}+\frac{1}{3}\frac{3!}{s^4}\,,\quad L\{y\}=\frac{1}{24}\frac{4!}{s^5}-\frac{1}{3}\frac{3!}{s^4}$$

so that

$$x=8+\frac{1}{24}t^4+\frac{1}{3}t^3 \quad \text{and} \quad y=\frac{1}{24}t^4-\frac{1}{3}t^3$$

4) In this Problem, use the definition of the Laplace transform to find $L\{f(t)\}$

$$f(t)=\begin{cases} t, & 0\le t<1 \\ 2-t, & t\ge 1\end{cases}$$ (Problem 1, page 232)

ANS $L\{f(t)\}=\int_0^1 te^{-st}dt+\int_1^\infty (2-t)e^{-st}dt=\frac{1}{s^2}-\frac{2}{s^2}e^{-s}$

5) Solve $L^{-1}\{\frac{s}{s^2-10s+29}\}$ (Problem 17, page 232)

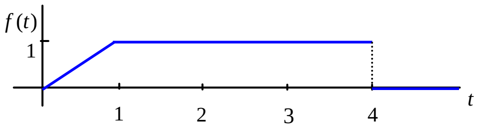
ANS $L^{-1}\{\frac{s}{s^2-10s+29}\}=L^{-1}\{\frac{s-5}{(s-5)^2+2^2}+\frac{5}{2}\frac{2}{(s-5)^2+2^2}\}=e^{5t}\cos 2t+\frac{5}{2}e^{5t}\sin 2t$

6) Solve $L^{-1}\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\}$ (Problem 19, page 232)

ANS $L^{-1}\{\frac{s+\pi}{s^2+\pi^2}e^{-s}\}=L^{-1}\{\frac{s}{s^2+\pi^2}e^{-s}+\frac{\pi}{s^2+\pi^2}e^{-s}\}=\cos\pi(t-1)u(t-1)$

$+\sin\pi(t-1)u(t-1)$

7) In this Problem, express f in terms of unit step functions. Find $L\{f(t)\}$ and $L\{e^t f(t)\}$ (Problem 29, page 233)



ANS $f(t)=t-[(t-1)+1]u(t-1)+u(t-1)-u(t-4)=t-(t-1)u(t-1)-u(t-4)$

$$L\{f(t)\}=\frac{1}{s^2}-\frac{1}{s^2}e^{-s}-\frac{1}{s}e^{-4s},\ L\{e^t f(t)\}=\frac{1}{(s-1)^2}-\frac{1}{(s-1)^2}e^{-(s-1)}-\frac{1}{s-1}e^{-4(s-1)}$$

8) In this Problem, use the Laplace transform to solve the given equation

$y'' + 6y' + 5y = t - t\,u(t-2); \quad y(0) = 1,\ y'(0) = 0$ (Problem 35, page 233)

ANS Taking the Laplace transform of the given differential equation we obtain

$$L\{y\} = \frac{s^3 + 6s^2 + 1}{s^2(s+1)(s+5)} - \frac{1}{s^2(s+1)(s+5)}e^{-2s} - \frac{2}{s(s+1)(s+5)}e^{-2s}$$

$$= -\frac{6}{25}.\frac{1}{s} + \frac{1}{5}.\frac{1}{s^2} + \frac{3}{2}.\frac{1}{s+1} - \frac{13}{50}.\frac{1}{s+5}$$

$$-(-\frac{6}{25}.\frac{1}{s} + \frac{1}{5}.\frac{1}{s^2} + \frac{1}{4}.\frac{1}{s+1} - \frac{1}{100}.\frac{1}{s+5})e^{-2s}$$

$$-(\frac{2}{5}.\frac{1}{s} - \frac{1}{2}.\frac{1}{s+1} + \frac{1}{10}.\frac{1}{s+5})e^{-2s}$$

so that

$$y = -\frac{6}{25} + \frac{1}{5}t^2 + \frac{3}{2}e^{-t} - \frac{13}{50}e^{-5t} - \frac{4}{25}u(t-2) - \frac{1}{5}(t-2)^2 u(t-2)$$

$$+\frac{1}{4}e^{-(t-2)}u(t-2) - \frac{9}{100}e^{-5(t-2)}u(t-2)$$