Review exercise 1s

1) Proceed as in Example 6 (page 519) to evaluate the given line integral

$$\oint_c \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}$$

where C is the ellipse $x^2 + 4y^2 = 4$ (Problem 25, page 521)

ANS We first observe that $P_y = (y^4 - 3x^2y^2)/(x^2+y^2)^3 = Q_x$ Letting C' be the circle

$$x^2 + y^2 = \frac{1}{4}$$

we have $\oint_C \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-y^3dx + xy^2dy}{(x^2+y^2)^2}$

$$x = \frac{1}{4}\cos t,\ dx = -\frac{1}{4}\sin t dt$$
$$y = \frac{1}{4}\sin t,\ dy = \frac{1}{4}\cos t dt$$

$$= \int_0^{2\pi} \frac{-\frac{1}{64}\sin^3 t(-\frac{1}{4}\sin t dt) + \frac{1}{4}\cos t(\frac{1}{16}\sin^2 t)(\frac{1}{4}\cos t dt)}{1/256}$$

$$= \int_0^{2\pi} (\sin^4 t + \sin^2 t\cos^2 t)dt = \int_0^{2\pi} (\sin^4 t + (\sin^2 t - \sin^4 t)dt$$

$$= \int_0^{2\pi} \sin^2 t dt = (\frac{1}{2}t - \frac{1}{4}\sin 2t)\Big|_0^{2\pi} = \pi$$

2) Evaluate $\iint_S (3z^2 + 4yz)\, dS$, where S is the portion of the plane $x + 2y + 3z = 6$ in the first octant. Use the portion of S onto the coordinate plane indicated in the given plane.

(Problem 25 and 26, page 528)

ANS 25 Write the equation of the surface as $y = \frac{1}{2}(6 - x - 3z)$

$$y_x = -\frac{1}{2},\quad y_z = -\frac{3}{2};\quad dS = \sqrt{1 + 1/4 + 9/4}\,dA = \frac{\sqrt{14}}{2}dA$$

$$\iint_S (3z^2 + 4yz)dS = \int_0^2\int_0^{6-3z}[3z^2 + 4z\frac{1}{2}(6 - x - 3z)]\frac{\sqrt{14}}{2}dxdz$$

$$= \frac{\sqrt{14}}{2}\int_0^2 [3z^2x - z(6 - x - 3z)^2]\Big|_0^{6-3z} dz$$

$$= \frac{\sqrt{14}}{2}\int_0^2 (36z - 18z^2)dz = \frac{\sqrt{14}}{2}(18z^2 - 6z^3)\Big|_0^2 = 12\sqrt{14}$$

ANS 26 Write the equation of the surface as $x = 6 - 2y - 3z$. Then

$$x_y = -2,\quad x_z = -3;\quad dS = \sqrt{1+4+9}\,dA = \sqrt{14}\,dA$$

$$\iint_S (3z^2 + 4yz)dS = \int_0^2\int_0^{3-3z/2}[3z^2 + 4yz]\sqrt{14}\,dydz$$

$$= \sqrt{14}\int_0^2 [3yz + 2y^2 z]\Big|_0^{3-3z/2}\, dz = \sqrt{14}\int_0^2 [9z(1-\frac{z}{2}) + 18z(1-\frac{z}{2})^2]dz$$

$$= \sqrt{14}(\frac{27}{2}z^2 - \frac{15}{2}z^3 + \frac{9}{8}z^4)\Big|_0^2 = 2\sqrt{14}$$

3) Use Stokes' theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$. Assume C is oriented counterclockwise as viewed from above. $\vec{F} = xy\vec{i} + 2yz\vec{j} + xz\vec{k}$. ; C the boundary of the plane $z = 1 - y$ shown in Figure 9.121(page 534) (Problem 9, page 534)

ANS $curl\,\overline{F} = (-3x^2 - 3y^2)\vec{k}$ A unit vector normal to the plane is $\vec{n} = (\vec{i} + \vec{j} + \vec{k})/\sqrt{3}$

From $z = 1 - x - y$, we have $z_x = z_y = -1$ and $dS = \sqrt{3}dA$

Then, using polar coordinates,

$$\oint_C \overline{F}\cdot d\vec{r} = \iint_S (curl\ \overline{F})\cdot \vec{n}dS = \iint_R (-\sqrt{3}x^2 - \sqrt{3}y^2)\sqrt{3}dA$$

$$= 3\iint_R (-x^2 - y^2)dA = 3\int_0^{2\pi}\int_0^1 (-r^2)rdrd\theta = 3\int_0^{2\pi} -\frac{1}{4}r^4\Big|_0^1\, d\theta = 3\int_0^{2\pi} -\frac{1}{4}d\theta = -\frac{3\pi}{2}$$

4) Use triple integrals and cylindrical coordinates to find the volume of the solid that is bounded by the graphs of the given equations.
$x^2 + y^2 = 4$, $x^2 + y^2 + z^2 = 16$, $z = 0$ (Problem 51, page 545)

ANS The equations are $r^2 = 4$, $r^2 + z^2 = 16$, and $z = 0$.

$$V = \int_0^{2\pi}\int_0^2\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{16-r^2}\,dr\,d\theta = \int_0^{2\pi}\frac{1}{3}(64 - 24\sqrt{3})d\theta$$

$$= \frac{2\pi}{3}(64 - 24\sqrt{3})$$

5) Use triple integrals and spherical coordinates to find the volume of the solid that is bounded by the graphs of the given equations.

$z = \sqrt{x^2 + y^2}$, $x^2 + y^2 + z^2 = 9$ (Problem 75, page 545)

ANS The equations are $\phi = \pi/4$ and $\rho = 3$.

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} 9\sin\phi\, d\phi\, d\theta = -9\int_0^{2\pi}(\frac{\sqrt{2}}{2} - 1)d\theta$$

$$= 9\pi(2 - \sqrt{2})$$

6) Consider the function $\vec{f}(x, y) = x^2y^4$. At $(1,1)$ what is:
(a) The rate of change of $\vec{f}$ in the direction of $\vec{i}$?
(b) The rate of change of $\vec{f}$ in the direction of $\vec{i} - \vec{j}$?
(c) The rate of change of $\vec{f}$ in the direction of $\vec{j}$? (Problem 29, page 559)

ANS $f_x = 2xy^4$, $f_y = 4x^2y^3$.

(a) $u = \vec{i}$, $D_u(1,1) = f_x(1,1) = 2$

(b) $u = (\vec{i} - \vec{j})/\sqrt{2}$, $D_u(1,1) = -2/\sqrt{2}$

(c) $u = \vec{j}$, $D_u(1,1) = f_y(1,1) = 4$

7) Find the volume of the solid shown in Figure 9.152 (page 559)

(Problem 38, page 559)

ANS $V = \int_0^{2\pi}\int_0^{\pi/6}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/6} (\frac{8}{3}\sin\phi - \frac{1}{3}\sin\phi) d\phi \, d\theta$

$$= \frac{7}{3}\int_0^{2\pi} (-\frac{\sqrt{3}}{2} + 1) d\theta = \frac{7\pi}{3}(2 - \sqrt{3})$$

8) Find the indicated expression for the vector field $\vec{F} = x^2y\vec{i} + xy^2\vec{j} + 2xyz\vec{k}$

(a) $\nabla \cdot \vec{F}$, (b) $\nabla \times \vec{F}$, (c) $\nabla \cdot (\nabla \times \vec{F})$, (d) $\nabla(\nabla \cdot \vec{F})$ (Problem 39~42, page 559)

ANS (a) $2xy + 2xy + 2xy = 6xy$

(b) $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xy^2 & 2xyz \end{vmatrix} = 2xz\vec{i} - 2yz\vec{j} + (y^2 - x^2)\vec{k}$

(c) $\frac{\partial}{\partial x}(2xz) - \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(y^2 - x^2) = 0$

(d) $\nabla(6xy) = 6y\vec{i} + 6x\vec{j}$

9) If $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, use the divergence theorem to evaluate $\iint_S (\vec{F} \cdot \vec{n}) \, dS$, where S is the surface of the region bounded by $x^2 + y^2 = 1$, $z = 0$, $z = 1$.

(Problem 59, page 560)

ANS $div\, \vec{f} = 1 + 1 + 1 = 3$;

$$\iint_S \vec{F} \bullet \vec{n} \, dS = \iiint_D div\, \vec{F} \, dV = \iiint_D 3 \, dV = 3 \times (volume\ of\ D) = 3\pi$$