Taylor Series For Functions of One Variable

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + \ldots$$

Taylor Series For Functions of Two Variables

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

$$+\frac{1}{2!}\left\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right\} + \ldots$$

For the functions of two variables $f(x+h\cos\theta, y+h\sin\theta)$

Let's consider Taylor Series for $f(x+h\cos\theta, y+h\sin\theta)$ about (x,y)

$$f(x+h\cos\theta, y+h\sin\theta) =$$

➔ $$f(x,y) + (h\cos\theta) f_x(x,y) + (h\sin\theta) f_y(x,y)$$

$$+\frac{1}{2!}\left\{\begin{array}{l}(h\cos\theta)^2 f_{xx}(x,y) + 2(h\cos\theta)(h\sin\theta) f_{xy}(x,y) \\ +(h\sin\theta)^2 f_{yy}(x,y)\end{array}\right\} + \ldots$$

$$f(x+h\cos\theta, y+h\sin\theta) - f(x,y) =$$

➔ $$h\left[(\cos\theta) f_x(x,y) + (\sin\theta) f_y(x,y)\right]$$

$$+\frac{h^2}{2!}\left[\begin{array}{l}(\cos\theta)^2 f_{xx}(x,y) + 2(\cos\theta)(\sin\theta) f_{xy}(x,y) \\ +(\sin\theta)^2 f_{yy}(x,y)\end{array}\right] + O(h^3)$$

where $O(h^3)$ denotes the remainder with degree ≥ 3.

The Gradient of a function

We define a vector valued function called the gradient as

$$\nabla f(x,y) = grad\ f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \langle f_x, f_y \rangle;$$

$$\nabla F(x,y,z) = grad\ F = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = \langle f_x, f_y, f_z \rangle$$

Definition 9.5 Directional Derivative

The directional derivative of $z = f(x,y)$ in the direction of a unit vector

$\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ is $D_{\vec{u}} f(x,y) = \lim_{h\to 0}\dfrac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h}$

Theorem 9.6 Computing a Directional Derivative

If $z = f(x,y)$ is a differentiable function of x and y and $\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$, then

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h}$$
$$= \nabla f(x,y)\cdot\vec{u}$$

Proof:

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h}$$

$$= \lim_{h\to 0}\left\{\begin{array}{l} \left[(\cos\theta) f_x(x,y) + (\sin\theta) f_y(x,y)\right] \\ + \dfrac{h}{2!}\left[\begin{array}{l}(\cos\theta)^2 f_{xx}(x,y) + 2(\cos\theta)(\sin\theta) f_{xy}(x,y) \\ + (\sin\theta)^2 f_{yy}(x,y)\end{array}\right] \\ + O(h^2)\end{array}\right\}$$

$$= (\cos\theta) f_x(x,y) + (\sin\theta) f_y(x,y)$$

$$= \left[f_x(x,y)\vec{i} + f_y(x,y)\vec{j}\right]\cdot(\cos\theta\,\vec{i} + \sin\theta\,\vec{j})$$
$$= \nabla f(x,y)\cdot\vec{u}$$
$$= comp_{\vec{u}} \nabla f$$

Similarly,

For a function $w = F(x,y,z)$ the directional derivative is defined by

$$D_{\vec{u}} F(x,y,z) = \lim_{h\to 0} \frac{F(x+h\cos\alpha, y+h\cos\beta, z+h\cos\gamma) - F(x,y,z)}{h}$$
$$= \nabla F(x,y,z)\cdot\vec{u}$$

where α,β,γ are the direction angles of the vector $\vec{u}$ relative to the positive x,y,z axes, respectively.

Maximum Value of the Directional Derivative

$$D_{\vec{u}} f = \|\nabla f\|\,\|\vec{u}\|\cos\phi = \|\nabla f\|\cos\phi,\quad (\|\vec{u}\| = 1)$$

where ϕ is the angle between ∇f and $\vec{u}$

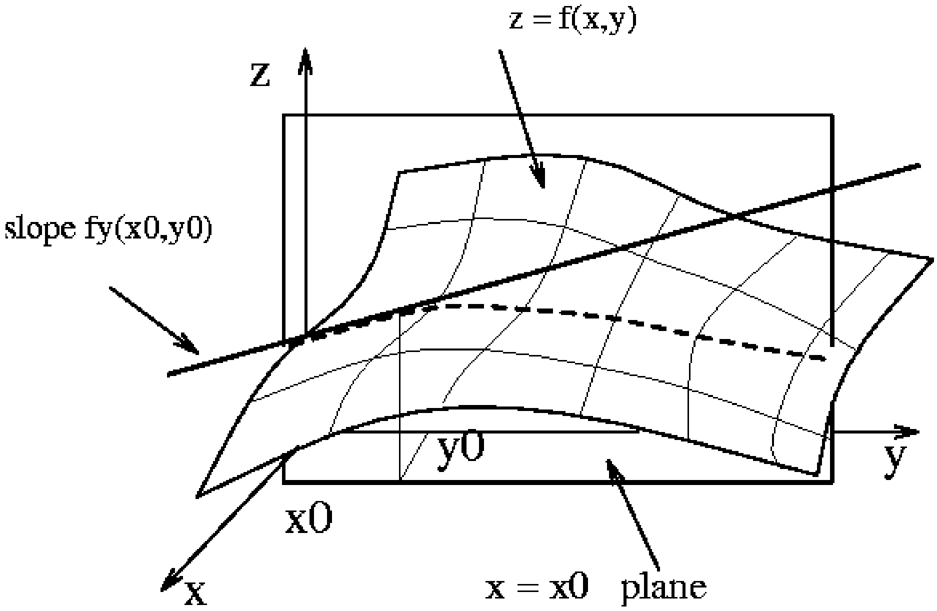
➔ $0 \le \phi \le \pi$ ==> $-1 \le \cos\phi \le 1$

➔ $-\|\nabla f\| \le \|\nabla f\|\cos\phi \le \|\nabla f\|$

➔ $-\|\nabla f\| \le D_u f \le \|\nabla f\|$

$D_{\vec{u}} f = \|\nabla f\|$ ← $\vec{u}$ has the same direction as ∇f.

$D_{\vec{u}} f = -\|\nabla f\|$ ← $\vec{u}$ and ∇f have opposite direction.



(From http://omega.albany.edu:8008/calc3/directional-derivatives-dir/define-m2h.html)

The partial derivative $f_y(x_0, y_0)$ is a special case of a directional derivative.

Consider a smooth function of two variables: $z = f(x, y)$
the partial derivative of $z = f(x, y)$ with respect to y at the point (x_0, y_0) is denoted by $f_y(x_0, y_0)$. Geometrically the number $f_y(x_0, y_0)$ is the slope of the tangent line at the point (x_0, y_0, z_0) to a curve on the surface $z = f(x, y)$. This curve is the intersection of the plane $x = x_0$ with the surface $z = f(x, y)$.

$$f_y(x_0, y_0) = D_{\vec{j}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{j}$$

Also,

$$f_x(x_0, y_0) = D_{\vec{i}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{i}$$