Engineering Mathematics II---Quiz-3s

April 28, 2006

1) Consider the function $f = x^2 y^4$. At $(1,1)$ what is:

(1) The rate of change of $f = x^2 y^4$ in the direction of $\vec{i}$? (5 scores)

ANS $f_x = 2xy^4$, $f_y = 4x^2 y^3$

$u = \vec{i}$, $D_u(1,1) = f_x(1,1) = 2$

(2) The rate of change of $f = x^2 y^4$ in the direction of $\vec{i} + \vec{j}$? (5 scores)

ANS $f_x = 2xy^4$, $f_y = 4x^2 y^3$

$u = (\vec{i} + \vec{j})/\sqrt{2}$, $D_u(1,1) = 6/\sqrt{2}$

2) Find the indicated expression for the vector field $\vec{F} = x^2 y\vec{i} + 2xyz\vec{k}$

(1) $\nabla \cdot \vec{F}$, (2) $\nabla \times \vec{F}$, (3) $\nabla \cdot (\nabla \times \vec{F})$, (4) $\nabla(\nabla \cdot \vec{F})$ (15 scores)

ANS (1) $4xy$

(2) $2xz\vec{i} - 2yz\vec{j} - x^2\vec{k}$

(3) 0

(4) $4y\vec{i} + 4x\vec{j}$

3) $\vec{F} = \dfrac{y}{x^2+y^2}\vec{i} - \dfrac{x}{x^2+y^2}\vec{j}$; Evaluate $\oint_C \vec{F} \cdot dr$

(1)C is shown as Fig1(a). (Hint: Using direct integral) (5 scores)

ANS $x = a\cos\theta$, $dx = -a\sin\theta d\theta$, $y = a\sin\theta$, $dy = a\cos\theta d\theta$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy = \int_0^{2\pi} \frac{a\sin\theta}{a^2}(-a\sin\theta d\theta) + \frac{-a\cos\theta}{a^2}(a\cos\theta d\theta)$$

$$= -\int_0^{2\pi}(\sin^2\theta + \cos^2\theta)d\theta = -2\pi$$

(2)C is shown as Fig1(b). (Hint: Using Green's theorem) (5 scores)

ANS $P = \dfrac{y}{x^2+y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$, $Q = \dfrac{-x}{x^2+y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = 0$$

(3) C is shown as Fig1(c). (Hint: Using Green's theorem) (5 scores)

ANS $\oint_C \vec{F}\cdot d\vec{r} = \int_{C'} \frac{y}{x^2+y^2}dx + \frac{-x}{x^2+y^2}dy = -2\pi$

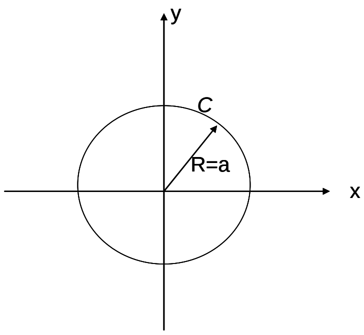

Fig. 1(a)

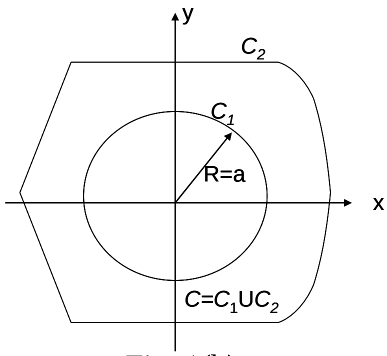

Fig. 1(b)

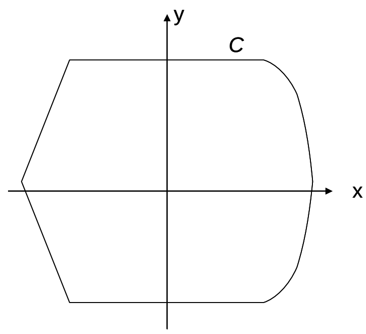

Fig. 1(c)

4) Suppose $\vec{r}(t) = 2t\vec{i} + (t^3 - 2t)\vec{j} + (t^2 - 5t)\vec{k}$ is the position vector of a moving particle. What are its speed, velocity, acceleration, curvature and tangent line at the point (0,0,0) ? (15 scores)

ANS $\vec{r}(t) = 2t\vec{i} + (t^3 - 2t)\vec{j} + (t^2 - 5t)\vec{k}$

$\vec{v}(t) = 2\vec{i} + (3t^2 - 2)\vec{j} + (2t - 5)\vec{k}$

$\vec{a}(t) = 6t\vec{j} + 2\vec{k}$

Speed: $\left|\vec{v}(0)\right| = \sqrt{(2)^2 + (-2)^2 + (-5)^2} = \sqrt{33}$

Velocity: $\vec{v}(0) = 2\vec{i} - 2\vec{j} - 5\vec{k}$

Acceleration: $\vec{a}(0) = 2\vec{k}$

Curvature: $\kappa = \frac{\left|\vec{v}(0)\times\vec{a}(0)\right|}{\left|\vec{v}(0)\right|^3} = \frac{\left|-4\vec{i} - 4\vec{j}\right|}{\sqrt{33}^3} = \frac{4}{33}\sqrt{\frac{2}{33}}$

Tangent line through $(0,0,0)$: $x = 0 + 2t,\ y = 0 + (-2)t,\ z = 0 + (-5)t$

5) S is the portion of the plane $x + 2y + 3z = 12$ in the first octant.

(1) find the area of S (5 scores)

ANS $z = f(x,y) = \frac{12 - x - 2y}{3}$

$$f_x(x,y) = -\frac{1}{3} \;\; , f_y(x,y) = -\frac{2}{3}, \;\; dS = \sqrt{1+f_x^{\,2}+f_y^{\,2}} = \sqrt{\frac{14}{9}}dA$$

$$A = \int_0^{12}\int_0^{\frac{12-x}{2}} \sqrt{\frac{14}{9}}dydx = \sqrt{\frac{14}{9}}\int_0^{12}\frac{12-x}{2}dx = \frac{1}{2}\sqrt{\frac{14}{9}}(12x-\frac{1}{2}x^2)\Big|_0^{12} = 12\sqrt{14}$$

(2) find the upper unit normal of S (5 scores)

ANS $g(x,y,z) = x+2y+3z = 12$, $\vec{n} = \frac{\nabla_g}{|\nabla_g|} = \frac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}}$

(3) Evaluate $\iint_S (3z^2+4yz)\,dS$ (10 scores)

ANS Write the equation of the surface as $y = \frac{1}{2}(12-x-3z)$

$$y_x = -\frac{1}{2}, \;\; y_z = -\frac{3}{2}; \;\; dS = \sqrt{1+1/4+9/4}dA = \frac{\sqrt{14}}{2}dA$$

$$\iint_S (3z^2+4yz)dS = \int_0^4\int_0^{12-3z}[3z^2+4z\frac{1}{2}(12-x-3z)]\frac{\sqrt{14}}{2}dxdz$$

$$= \frac{\sqrt{14}}{2}\int_0^4 [3z^2x - z(12-x-3z)^2]\Big|_0^{12-3z} dz$$

$$= \frac{\sqrt{14}}{2}\int_0^4 (144z^2-36z^3)dz = \frac{\sqrt{14}}{2}(72z^3-12z^4)\Big|_0^4 = 192\sqrt{14}$$

6) If S is the surface of the region bounded by $x^2+y^2=9$, $z=\sqrt{16-x^2-y^2}$,

$z=0$. $\vec{F} = -y^3\vec{i} - x^3\vec{j} + z^3\vec{k}$

(1) find the volume of the solid bounded by $x^2+y^2=9$, $z=\sqrt{16-x^2-y^2}$,

$z=0$. (10 scores)

ANS $V = \iiint_D dxdydz = \int_0^{2\pi}\int_0^3\int_0^{\sqrt{16-r^2}} rdzdrd\theta = \int_0^{2\pi}\int_0^3 r\sqrt{16-r^2}drd\theta$

$$= \int_0^{2\pi} -\frac{1}{3}(7\sqrt{7}-64)d\theta = -\frac{14}{3}\sqrt{7}\pi + \frac{128}{3}\pi$$

(2) use the divergence theorem to find the outward flux $\iint_S \left(\vec{F}\cdot\vec{n}\right) dS$ (15 scores)

ANS $div\,\vec{F} = \frac{\partial(-y^3)}{\partial x} + \frac{\partial(-x^3)}{\partial y} + \frac{\partial(z^3)}{\partial z} = 3z^2$

Using cylindrical coordinates,

$$\iint_S \vec{F}\cdot\vec{n}dS = \iiint_D div\,\vec{F}dV = \int_0^{2\pi}\int_0^3\int_0^{\sqrt{16-r^2}} 3z^2 rdzdrd\theta$$

$$= \int_0^{2\pi} \int_0^3 rz^3 \Big|_0^{\sqrt{16-r^2}} drd\theta = \int_0^{2\pi} \int_0^3 r(16-r^2)^{3/2} drd\theta$$

$$= \int_0^{2\pi} -\frac{1}{5}(16-r^2)^{5/2} \Big|_0^3 d\theta = \int_0^{2\pi} -\frac{1}{5}(7^{5/2} - 4^5) d\theta$$

$$= \int_0^{2\pi} \frac{1}{5}(1024 - 49\sqrt{7}) d\theta = \frac{2\pi}{5}(1024 - 49\sqrt{7})$$