Tangent Vector

For a curve C with position vector $\vec{r}(t)$, we know that $\vec{r}\,'(t)$ is a tangent vector to the curve C.

$$\vec{r}\,'(t) = \frac{d\vec{r}(t)}{dt} = \frac{d\vec{r}(t)}{ds}\frac{ds}{dt}$$

$$\frac{ds}{dt} = \left\|\frac{d\vec{r}(t)}{dt}\right\| = \left\|\vec{r}\,'(t)\right\|$$

$$\Rightarrow \frac{d\vec{r}(t)}{ds} = \frac{d\vec{r}(t)/dt}{ds/dt} = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} = \vec{T}$$

the unit tangent vector $\vec{T}(t)$ is defined by

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} = \frac{\vec{r}\,'(t)}{s'} = \frac{d\vec{r}}{ds}$$

where t is a parameterization variable, s is the arc length.

We can also say that a body or a particle moves along the curve C parameterized by three equations $x = f(t), y = g(t), z = h(t)$, $a \le t \le b$. So that its position at the time t is given by the vector function $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$.

If f, g, h have second derivatives, then the vectors

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \left\langle \frac{df(t)}{dt}, \frac{dg(t)}{dt}, \frac{dh(t)}{dt}\right\rangle = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

$$\vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2} = \left\langle \frac{d^2 f(t)}{dt^2}, \frac{d^2 g(t)}{dt^2}, \frac{d^2 h(t)}{dt^2}\right\rangle = f''(t)\vec{i} + g''(t)\vec{j} + h''(t)\vec{k}$$

are called the velocity and acceleration of the particle, respectively.

And the speed of the particle is the scalar function $\left\|\vec{v}(t)\right\|$ with

$$\left\|\vec{v}(t)\right\| = \left\|\frac{d\vec{r}(t)}{dt}\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}$$

$$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|} = \frac{\vec{v}(t)}{\left\|\vec{v}(t)\right\|} = \frac{f'(t) + g'(t) + h'(t)}{\sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}}$$

Length of a Space Curve

s is the arc length.

$$\frac{ds}{dt} = \left\| \frac{d\vec{r}(t)}{dt} \right\| = \left\| \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} \right\|$$
$$= \left\| \frac{df}{dt}\vec{i} + \frac{dg}{dt}\vec{j} + \frac{dh}{dt}\vec{k} \right\|$$
$$= \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}$$

➔ $ds = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt$ **or** $ds = \left\| \frac{d\vec{r}(t)}{dt} \right\| dt = \left\| \vec{r}^{\,'}(t) \right\| dt$

The length of the smooth curve traced by $\vec{r}(t)$ is given by

$$s = \int_a^b ds = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}\, dt = \int_a^b \left\| \vec{r}^{\,'}(t) \right\| dt$$

Curvature

Definition: Curvature

Let $\vec{r}(t)$ be a vector function defining a smooth curve C. If s is the arc length parameter and $\vec{T} = d\vec{r} / ds$ is the unit tangent vector, then the curvature of C at a point is

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

➔measuring how fast a curve bends at any spot.

If the curve isexpressed in terms of a general parameter t

➔ $\frac{d\vec{T}}{dt} = \frac{d\vec{T}/ds}{ds/dt}$ ➔ $\frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt}$

➔ $\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| = \frac{\left\| \vec{T}^{\,'}(t) \right\|}{\left\| \vec{r}^{\,'}(t) \right\|}$ (**Note**: $\frac{ds}{dt} = \left\| \frac{d\vec{r}(t)}{dt} \right\| = \left\| \vec{r}^{\,'}(t) \right\|$)

Example 1 (page 460)

Find the curvature of a circle radius a

Sol: A circle can be described by the vector function $\vec{r}(t) = a\cos(t)\vec{i} + a\sin(t)\vec{j}$

➔ $\dfrac{d\vec{r}(t)}{dt} = -a\sin(t)\vec{i} + a\cos(t)\vec{j}$

➔ $\left\|\dfrac{d\vec{r}(t)}{dt}\right\| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2} = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = a$

➔ $\vec{T} = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = -\sin(t)\vec{i} + \cos(t)\vec{j}$ ➔ $\vec{T}' = -\cos(t)\vec{i} - \sin(t)\vec{j}$

➔ $\kappa = \dfrac{\|T'(t)\|}{\|\vec{r}'(t)\|} = \dfrac{\sqrt{\cos^2(t) + \sin^2(t)}}{a} = \dfrac{1}{a}$

➔ $a = \dfrac{1}{\kappa}$ is called the **radius of curvature**

➔A circle with a small radius (➔ large curvature) curves more than one with a large radius (➔small curvature).