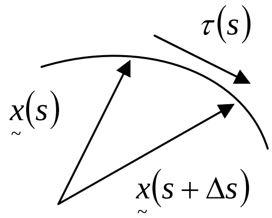
Frenet formula Oct,13,2004

Step 1. Give a space curve of $(x(t), y(t), z(t))$

Step 2. Path coordinate $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

Spet 3. $((x(t), y(t), z(t))) \to X(s) = ((\tilde{x}(s), \tilde{y}(s), \tilde{z}(s)))$ transform t to s

Step 4. Tangent vector $\tau(s) = \dfrac{\dot{X}(s)}{\left\|\dot{X}(s)\right\|}$



Step 5. $\underset{\sim}{\tau} \cdot \underset{\sim}{\tau} = 1 \Rightarrow \underset{\sim}{\tau} \cdot \underset{\sim}{\dot{\tau}} = 0 \quad \dot{\tau} \perp \tau$ (Note that $|\tau| = |\nu| = |\beta| = 1$)

Choose $\underset{\sim}{\nu} = \rho \dot{\tau}$ (for simplicity)

Step 6. Why ρ is the radius of curvature

$$\tau(s) \cdot \tau(s+\Delta s) = |\tau(s)||\tau(s+\Delta s)|\cos(d\theta) = \cos(d\theta)$$

$$\tau(s) \cdot \tau(s+\Delta s) = \tau(s) \cdot \left[\tau(s) + \frac{1}{1!}\dot{\tau}(s)(ds) + \frac{1}{2!}\ddot{\tau}(s)(ds)^2 + \frac{1}{3!}\dddot{\tau}(s)(ds)^3 + \cdots\right]$$

$$= 1 + \tau(s) \cdot \dot{\tau}(s)(ds) + \frac{1}{2}\tau(s) \cdot \ddot{\tau}(s)(ds)^2 + \frac{1}{6}\tau(s) \cdot \dddot{\tau}(s)(ds)^3 + \cdots$$

$$= 1 + \frac{1}{2}\tau(s) \cdot \ddot{\tau}(s)(ds)^2 + \cdots$$

$$\cos(d\theta) = 1 - \frac{1}{2!}(d\theta)^2 + \frac{1}{4!}(d\theta)^4 - \frac{1}{6!}(d\theta)^6 + \cdots = 1 - \frac{1}{2!}(d\theta)^2 + \cdots$$

$$1 + \frac{1}{2}\tau(s) \cdot \ddot{\tau}(s)(ds)^2 = 1 - \frac{1}{2!}(d\theta)^2 \Rightarrow \quad \tau(s) \cdot \ddot{\tau}(s)(ds)^2 = -(d\theta)^2 \qquad (1)$$

$\tau \cdot \dot{\tau} = 0 \Rightarrow \dot{\tau} \cdot \dot{\tau} + \tau \cdot \ddot{\tau} = 0 \quad \therefore \tau \cdot \ddot{\tau} = -\dot{\tau} \cdot \dot{\tau}$ 代入(1)

$$-\dot{\tau} \cdot \dot{\tau}(ds)^2 = -(d\theta)^2 \Rightarrow |\dot{\tau}|^2 = \left(\frac{d\theta}{ds}\right)^2 = \frac{1}{\left(\frac{ds}{d\theta}\right)^2} = \frac{1}{\left(\frac{\rho d\theta}{d\theta}\right)^2} = \frac{1}{\rho^2} \quad \therefore \rho d\theta = ds$$

$$\therefore |\dot{\tau}| = \frac{1}{\rho} \ , \ \rho = \frac{ds}{d\theta}$$

Step 7. $\underset{\sim}{\beta} \cdot \underset{\sim}{\beta} = 1 \Rightarrow \underset{\sim}{\dot{\beta}} \cdot \underset{\sim}{\beta} = 0$

$\therefore \underset{\sim}{\dot{\beta}} = p\underset{\sim}{\tau} + q\underset{\sim}{\nu} \quad \Rightarrow$

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$$\underset{\sim}{\tau}\cdot\underset{\sim}{\beta}=0$$

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}+\underset{\sim}{\dot{\tau}}\cdot\underset{\sim}{\beta}=0$$

$$\underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}+\frac{1}{\rho}\underset{\sim}{\nu}\cdot\underset{\sim}{\beta}=0$$

$$\therefore \underset{\sim}{\tau}\cdot\underset{\sim}{\dot{\beta}}=0$$

$$\underset{\sim}{\dot{\beta}}\cdot\tau=p\underset{\sim}{\tau}\cdot\underset{\sim}{\tau}+q\underset{\sim}{\nu}\cdot\underset{\sim}{\tau}$$

$$0=p$$

$\therefore set\ \dot{\beta}=-\frac{1}{\sigma}\nu$ (for simplicity)

Step 8. $\underset{\sim}{\nu}=\underset{\sim}{\beta}\times\underset{\sim}{\tau}$

$$\underset{\sim}{\dot{\nu}}=\underset{\sim}{\dot{\beta}}\times\underset{\sim}{\tau}+\underset{\sim}{\beta}\times\underset{\sim}{\dot{\tau}}$$

$$=-\frac{1}{\sigma}\underset{\sim}{\nu}\times\underset{\sim}{\tau}+\underset{\sim}{\beta}\times\frac{1}{\rho}\underset{\sim}{\nu}$$

$$=\frac{1}{\sigma}\underset{\sim}{\beta}-\frac{1}{\rho}\underset{\sim}{\tau}$$

∴we have

$$\begin{Bmatrix}\dot{\tau}\\ \dot{\nu}\\ \dot{\beta}\end{Bmatrix}=\begin{bmatrix}0 & \frac{1}{\rho} & 0\\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma}\\ 0 & -\frac{1}{\sigma} & 0\end{bmatrix}\begin{Bmatrix}\tau\\ \nu\\ \beta\end{Bmatrix}\rightarrow \underset{\sim}{\dot{P}}=A\underset{\sim}{P}$$

where $A=\begin{bmatrix}0 & \frac{1}{\rho} & 0\\ -\frac{1}{\rho} & 0 & \frac{1}{\sigma}\\ 0 & -\frac{1}{\sigma} & 0\end{bmatrix}$ (反對稱)， $\underset{\sim}{P}=\begin{Bmatrix}\tau\\ \nu\\ \beta\end{Bmatrix}$