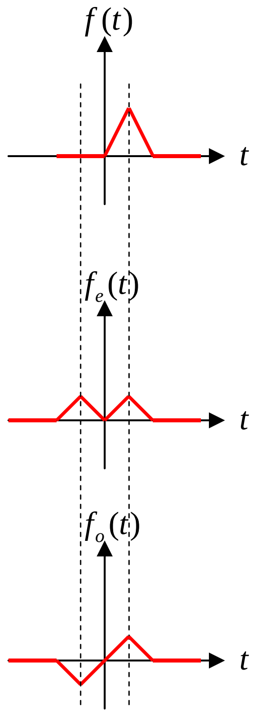
因果函數 Fourier 轉換特性



Let $f(t)=f_e(t)+f_o(t)$ and $F(\omega)=F_R(\omega)+iF_I(\omega)$

where $f_o(t)=f_e(t)\cdot\mathrm{sgn}(t)$, $f_e(t)=f_o(t)\cdot\mathrm{sgn}(t)$,

$$F(f_e(t))=F_R(\omega),\quad F(f_o(t))=iF_I(\omega),\ \text{and}\ \ F(\mathrm{sgn}(t))=S(\omega)=\frac{-2i}{\omega}$$

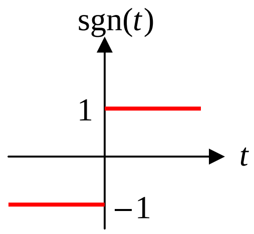
$$F(f(t)*g(t))=F(\omega)\cdot G(\omega)\qquad \Rightarrow F(F(\omega)*G(\omega))=4\pi^2 f(-t)\cdot g(-t)$$

(1) $F_R(\omega)=\int_{-\infty}^{\infty}\frac{F_I(u)}{\pi(\omega-u)}du$

$$f_e(t)=f_o(t)\cdot\mathrm{sgn}(t)=(-f_o(-t))\cdot(-\mathrm{sgn}(-t))=f_o(-t)\cdot\mathrm{sgn}(-t)$$

$$\begin{aligned}\Rightarrow F(iF_I(\omega)*S(\omega))&=2\pi f_o(-t)\cdot 2\pi\,\mathrm{sgn}(-t)\\&=2\pi\cdot 2\pi f_e(-t)\\&=2\pi F(F_R(\omega))\end{aligned}$$

$$\Rightarrow F_R(\omega)=\frac{1}{2\pi}(iF_I(\omega)*S(\omega))=\int_{-\infty}^{\infty}\frac{iF_I(u)}{2\pi}\cdot S(\omega-u)du=\int_{-\infty}^{\infty}\frac{iF_I(u)}{2\pi}\cdot\frac{-2i}{(\omega-u)}du=\int_{-\infty}^{\infty}\frac{F_I(u)}{\pi(\omega-u)}du$$



(2) $F_I(\omega)=\int_{-\infty}^{\infty}\frac{-F_R(u)}{\pi(\omega-u)}du$

$$f_o(t)=f_e(t)\cdot\mathrm{sgn}(t)=(f_e(-t))\cdot(-\mathrm{sgn}(-t))=-f_e(-t)\cdot\mathrm{sgn}(-t)$$

$$\begin{aligned}\Rightarrow F(F_R(\omega)*S(\omega))&=2\pi f_e(-t)\cdot 2\pi\,\mathrm{sgn}(-t)\\&=2\pi\cdot 2\pi\cdot f_o(-t)\\&=2\pi\cdot F(iF_I(\omega))\\&=2\pi i\cdot F(F_I(\omega))\end{aligned}$$

$$\Rightarrow F_I(\omega)=\frac{1}{2\pi i}(F_R(\omega)*S(\omega))=\int_{-\infty}^{\infty}\frac{F_R(u)}{2\pi i}\cdot S(\omega-u)du=\int_{-\infty}^{\infty}\frac{F_R(u)}{2\pi i}\cdot\frac{-2i}{(\omega-u)}du=-\int_{-\infty}^{\infty}\frac{F_R(u)}{\pi(\omega-u)}du$$