Professor C.-S. Liu
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May 25, 2004

Dear Professor Liu,

Thank you for agreeing to review Peter V. O' Neil's *Advanced Engineering Mathematics*. The development process is a long one, and requires that we collect information soon after a book is published in preparation for what improvements need to be made for the next edition. Your participation in this review panel will enable us to refine the manuscript for the next edition so that we can publish a textbook that address the needs of both instructors and students. Your feedback will be provided to the author anonymously, so please feel free to be completely candid. I would also appreciate constructive criticism so as to provide the author with concrete guidance on how to revise and improve the textbook.

The questions below should serve as guidelines to the kind of information that I am interested in obtaining. My questions tend to focus on the structural issues of the text but please do not feel constrained by them. Most importantly, I am interested in learning what you like about this textbook, what you feel needs to be changed or improved, and why you use this textbook. I would also appreciate any suggestions you have on ways to improve the presentation of the content, and will pass those comments on to the author. An expanding text box for your response follows each question. Please use these boxes to provide detailed and specific review comments.

I am offering honoraria of $300.00 USD for your feedback and need to have the completed review returned by June 14, 2004. I am also offering a bonus of $100USD for the submission of your review by this date.

I want to thank you once again for agreeing to review this textbook. If you have any questions please do not hesitate to contact me.

Sincerely,

Joanne Woods
Editorial Assistant
Thomson Engineering
1 866 349 2431 ext.3348
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Thomson Engineering

REVIEWER HONORARIUM SLIP

Thank you for agreeing to review. Please complete and return with your review to the email address below. This is a Word document so you may use as much space as you require.

Title of Project:	*Advanced Engineering Mathematics* by O' Neil
Reviewer:	Professor Chein-Shan Liu
Mailing Address:	csliu@mail.ntou.edu.tw
School:	National Taiwan Ocean University
Return to:	Joanne Woods, Editorial Assistant Email: Joanne.Woods@thomson.com
Honorarium:	**$300** (Payable upon receipt of completed review) Deadline: June 14, 2004

Note to reviewer:

Your comments will be given to the author(s) anonymously.
Upon publication we will, unless you indicate below, acknowledge you as one of several reviewers in the preface of the book.

_____ I choose not to be acknowledged as a reviewer in the preface of the book.

Upon publication, Thomson Engineering may choose to use your comments in our promotional material. Please indicate below if you wish to be contacted again prior to such use.

__**X**___ I prefer to be contacted prior to using my comments in the promotion of the text.

Name: C.-S. Liu (simply type in name)

Telephone #: 886-2-24622192 ext. 3252

Email: csliu@mail.ntou.edu.tw

Reviewer Profile:

What is the course name and number for which this text would be used in your school? (Please include course syllabus if possible).

The course names are Engineering Mathematics (I) and (II). There are two semesters Courses for the sophomore: one is B7212086 for Engineering Mathematics (I) and the other one is B7222086 for Engineering Mathematics (II).
In the course of Engineering Mathematics (I), the syllabus is divided into six main parts:
(1) First-Order Ordinary Differential Equations (Chapter 1),
(2) Second-Order Ordinary Differential Equations (Chapter 2),
(3) Higher-Order Ordinary Differential Equations and Systems of Differential Equations (Chapter 9),
(4) Qualitative Behavior of Systems of Nonlinear Differential Equations (Chapter 10),
(5) Laplace Transform (Chapter 3),
(6) Series Solutions (Chapter 4).

In the course of Engineering Mathematics (II), the syllabus is divided into six main parts:
(1) Linear Algebra (Chapters 6-8),
(2) Vector Space and Operations (Chapter 5),
(3) Vector Differential Calculus (Chapter 11),
(4) Vector Integral Calculus (Chapter 12),
(5) Partial Differential Equations (Chapters 16-18),
(6) Fourier Series and Transform (Chapters 13-14).

How long have you been teaching this course? Full time or part-time?

I had two years experience part-timely teaching this course when I was a student in Fung-Chia University. Now I have been teaching this course for four years as the full time professor of the Department of Mechanical and Mechatronic Engineering in National Taiwan Ocean University.

Who else teaches the course?

As I know Professor N.N. Huang of the Department of Mechanical and Mechatronic Engineering, Professor J.T. Chen of the Department of Harbor and River Engineering of NTOU, and Professor H.K. Hong of the Department of Civil Engineering of National Taiwan University all teach this course.

What is the annual enrollment?

About 130 students per year.

What is the name and author of the text currently in use?

Advanced Engineering mathematics, 5th ed., by P.V. O'Neil.

What do you like and/or dislike about the text currently in use?

In general, this textbook provides a rather comprehensive treatment of many topics of engineering mathematics, which is required for students before they take the professional courses in junior and senior years. From a mathematical point of view, this textbook must enhance the definitions and notations and give more precise and definite explanations.

This text is easy to follow for students.

According to the questionnaire I make for students, I find that most students can follow this textbook without difficulty.

What text did you use previous to your current text?

I never used other textbook.

What is your key adoption criteria (the most important characteristic for adoption)?

I adopt this text as our textbook on Engineering Mathematics, because its presentations are organized well.

Is adoption an individual or committee decision? If committee, who is the chair and/or other key people?

The adoption of this textbook is individual.

Would you like to do more reviewing for us in the future? If so, please provide the following information about other courses you instruct:

	Course Number	Course Name	Text/Author	Annual Enrollment
1)				
2)				

The other three courses I teach are: Plasticity Theory, Frictional Dynamics and Partial Differential Equations in the institute.

Do you have any interest in writing or adapting a text or supplement? In writing software? If yes, please indicate text/subject areas or area of specialty below:

1)

2)

3)

At this moment I focus on writing Scientific Papers, and have no plan to write text.

REVIEW QUESTIONS
O' Neil, *Advanced Engineering Mathematics*
Part A. Textbook Review

Organization. Please comment on the overall organization of this textbook. Please be as specific as possible in your answers. Are the topics organized effectively?

The overall arrangement of this textbook is rather smooth and it is organized effectively.

Is there a chapter(s), or topic(s) within the existing chapters that is missing, and should be included?

This textbook gives no relation between the Riccati equation in Section 1.6.3 with the second order homogeneous equation in Section 2.2.1. In Appendix A, I demonstrate such relation by the concept of integrating factor. The author may consider to include it into the new version. From this note the students may appreciate the relation of the first order nonlinear Riccati equation with second order linear equation.

In Section 5.3 of the topic of cross product of vectors, in addition to the standard operation of cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = (A_2B_3 - A_3B_2)\mathbf{i} + (A_3B_1 - A_1B_3)\mathbf{j} + (A_1B_2 - A_2B_1)\mathbf{k},$$

I taught the students with the permutation symbol:

$$\mathbf{A}\times\mathbf{B} = e_{ijk}\, A_j B_k \mathbf{e}_i ,$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are standard bases of R^3. In Appendix B the vectors operations of inner product, cross product and their compositions are included. As the students told me after learning such method, they were not afraid of the complex calculations of vectors anymore. In this sense the author may consider to rearrange the vector notations and bases (see also the comments in the later).

By the same token in Sections 11.4 and 11.5 of the topics of gradient, divergence and curl of scalar and vector fields, I taught the students with the following operations:

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i ,$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_i}{\partial x_i} ,$$

$$\nabla\times\mathbf{A} = e_{ijk}\,\frac{\partial A_k}{\partial x_j}\mathbf{e}_i .$$

In Appendix C the vectors differential calculus are included. From this appendix it is clear that many complex formulae about vectors differential calculus become easy to handle.

This textbook gives a detailed description of first order and second order differential equations, but does not mentions the higher order differential equations. In Chapter 9 the author may consider to add a section about transforming the higher order differential equations into a system of linear differential equations.

Are there any topics or chapters that you feel are unnecessary and could be deleted?

Part 8 in this textbook seems writing for the teachers not for the students. I doubt that how many students will read this part when they are busy to learn many new mathematical topics.

Should any of the material be reorganized? Please give your reasons for any suggested reorganization.

This textbook does not use the index notations for vectors and the Einstein summation convention for repeated indices. However, I think that it is important to let students learn these operations before they enter the institute. Therefore, the author may consider to write

$$\mathbf{F} = F_i \mathbf{e}_i , \quad \mathbf{G} = G_i \mathbf{e}_i$$

instead of the cumbersome ones

$$\mathbf{F} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k} , \quad \mathbf{G} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k}$$

as shown in page 216.

Approach. Please comment on the author's approach to the topics presented in the text, the need it intends to meet, and the features and benefits espoused by the author. Are these the types of things that appeal to you as a potential user of the text? What challenges do you face in teaching this course?

This textbook has been written primarily to facilitate the grasp of both the fundamental concepts and the techniques of engineering mathematics by the beginning student. The approach to many topics by the author is according to this principle. However, some more unified frameworks may be invoked; for example, the three integral theorems of Green, Gauss and Stokes can be written as

$$\int_{D} dw = \int_{\partial D} w ,$$

where w is a differential form in the space domain D, and ∂D is the boundary of D. By letting $w = P(x,y)dx + Q(x,y)dy$ be an one-form we have Green's Theorem:

$$\int_{\partial D} P(x,y)dx + Q(x,y)dy = \int_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

By letting $w = P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$ be a two-form we have Gauss' Theorem:

$$\int_{\partial D} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy = \int_{D} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz .$$

By letting $w = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ be an one-form we have Stokes' Theorem:

$$\int_{\partial D} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_D \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

In teaching the course of engineering mathematics there are two challenges for me: the physical motivations and the fundamental mathematical structures. If the author can further exposure these two features for most students many other teachers may be potential users of this textbook.

Level. Please evaluate the level of presentation in the chapters. Is the material written at a level that your students will understand?

This textbook has been written primarily to facilitate the grasp of both the fundamental concepts and the techniques of engineering mathematics for beginner students. In addition to its use as a beginning text on engineering mathematics, it should also serve as a reference for advanced students, especially in the engineering science fields, who may no longer learn more engineering mathematics in their future course.

Is the author paying attention to the audience, defining new terms when they are introduced, pacing the presentation so that the audience will be able to keep up and understand concepts as they are developed, and motivating the presentation with examples students can understand and relate to?

According to the questionnaire made on the students, the author has paying much attention to the audience defining new terms when they are introduced, pacing the presentation so that the audience is able to keep up and understand concepts as they are developed, and motivating the presentation with examples students can understand and relate to.

Presentation. Is the material written in a coherent fashion, with a clear sense of flow between topics and sections?

Most materials are written in a coherent fashion; however, smoothing the flow between topics and sections can be enhanced.

Is the abstract theoretical material balanced with concrete examples that students can understand and relate to?

Throughout this textbook, applications are interspersed with mathematical theories and development. Thus the study of applications to engineering problems is carried along with theories. In this way, insight can be easily developed, and examples and exercises also provide practices for students.

Is the material written in a precise manner without being overly academic so as to turn students off with its "dryness"?

Many topics of an advanced nature were omitted in this textbook, which by no means that this textbook is imprecise. A mathematical text should cover the fundamentals of mathematical analysis. The applications to engineering problems are limited to those with which most students will be expected to have some familiarity. In my opinion the materials presented in this textbook are not overly academic.

Example Problems. Please comment on the quality of the example problems. Are there a sufficient number of example problems to motivate and explicate the concepts presented in the textual presentation?

The more examples will make the textbook more valuable to a large number of engineering self learners. To carry out this objective, the author has included examples and exercises that illustrate many applications of theories to engineering problems. In addition to most common problems, there are many problems bringing out more difficult aspects of the subject. They serve the dual purpose of introducing the student to a deeper appreciation of the subject and indicating many applications which might not be apparent while first reading.

Does the author provide the students with a good problem-solving methodology in the examples? Specifically, does the author provide enough suggestions, hints, and solutions techniques so that the examples clarify the information?

According to the questionnaire made on the students, the author has provided the students with a good problem-solving methodology in the examples. Specifically, the author provides enough suggestions, hints, and solutions techniques so that the examples clarify the information.

Illustrations. Please comment on the number of illustrations that the author uses to reinforce the textual presentation.

It is believed that engineering mathematics should be considered as both a mathematical discipline and a language of physics. The close relation between these two branches of science that arises naturally is not difficult to see, but the ability to think in terms of engineering mathematics is an art that requires both insight and practice. It is the intent of the author to present the material in such a way that both objectives will result from a close study of this textbook. In this textbook the author provides a lot of illustrations to reinforce the textual presentation. However, in the Section 15.3 of Sturm-Liouville Theory and Eigenfunction Expansions a good illustration may be the Schrödinger eigenvalue problem:

$$-y''(x)+V(x)y(x)=\lambda y(x),\;\; y(\pm\infty)=0,$$

where $V(x)$ is the potential function, for example $V(x)=x^2/4$ for quantum harmonic oscillator. The above equation possesses infinite number real positive eigenvalues. Physically, the eigenvalues are the allowed energy levels of a particle in the potential

$V(x)$.

Homework Exercises. Please review the homework exercises. Are there a sufficient number included in the manuscript? Normally how many homework exercises would you expect to find at the end of each section or chapter?

Really, this textbook has provided too many exercises. In each section, 20-30 exercises are enough. Most exercises of the same sort can be deleted.

Do the homework exercises cover a good range of difficulty from simple "plug and chug" problems, to more complicated and challenging problems?

From the reflection of the questionnaire answered by students, the homework exercises cover a suitable range of difficulty of the problems.

Strengths and Weaknesses. How would you rate this textbook in comparison to the textbook you are currently using? Is this book presenting the material in a way that appeals to you?

Among this textbook and the other two I am reading, I think the most easy one is Zill, O' Neil is medium, and then the most difficult is Kreyszig. O' Neil locates at the middle position, and designing a middle course is just the main characteristic of this textbook. If the author can enhance its character by unifying many treatments as suggested in the above and in the Appendices, this textbook may appeal more teachers to use it as the textbook of engineering mathematics.

What have you felt this textbook' s strongest features are?

I think it is easier to follow than the textbook by Kreyszig and more difficult than Zill. It locates in a middle position, and taking a middle course is just the strongest feature of this textbook.

Conversely, what weaknesses must be corrected for the book to be acceptable to you?

Precise definitions of many concepts and notations can be enhanced, and the logical arrangement can be improved. In the Engineering Mathematics (I), the main course is Ordinary Differential Equations. So the arrangement of this textbook may consider this direction by putting Part III behind Part I.

Part B. About Your Course

Software. Do you use any computational software tools like MATLAB or Mathcad in presenting your course?

Up till now I have not used any computational software in the course of engineering mathematics.

Do you expect your students to use any computer-based tools to solve problems in your course?

I expect my students can use computer-based tools to solve problems in the course of engineering mathematics.

Do you want the textbook to have specific types of problems that require a computer for solution?

I think this point is important, if this textbook can provide some problems that require a computer for solution.

Assignments. Do you assign any examples or problems that would be considered open-ended design problems in your course? Do you want the textbook to have a selection of these types of problems?

I have never assigned such open-ended problems in the course. However, if this textbook can provide such type of problems, the teacher may use it in the course of engineering mathematics.

Ancillary Materials. What ancillary materials should we provide to help you use the book in your course?

In order to visualize phase portraits, solutions, direction fields, convergence of Fourier series, Gibbs phenomenon as well as filtering noise from signals some software should be designed and provided as ancillary materials.

Do you require, or would you use the complete detailed solutions to all homework exercises in the text in a printed Solutions Manual? In PDF format to allow posting of solutions on your Website?

The Solutions Manual can help teachers to select some suitable exercises to students. The teaching assistants may get great help from a complete Solutions Manual when they examine the school assignments. If there are PDF formats, teachers can post them on the Website.

Do you require, or would you use a CD with all figures from the examples and homework exercises to allow use of PowerPoint presentations in your class?

The PowerPoint presentations of the courses may be effective. If there is a CD including all Figures the teacher would use the PowerPoint to show the related courses.

What instructor ancillary materials does your current text provide?

The other texts do not provide any ancillary material up till now.

Internet Use. Please describe the use of the World Wide Web in your course?

Putting the whole course onto Website is the current trend in our university. In order to achieve this goal, teachers will expect that this textbook can provide PDF files of the course that can post on Website.

Any additional comments that you feel would help us improve this manuscript will be greatly appreciated.

More comments are relegated into Appendices A-C.

Thank you again for your time in completing this review.

Appendix A

We rewrite the Riccati differential equation

$$\frac{dy}{dt} = r(t) - 2q(t)y - p(t)y^2 \tag{A.1}$$

to be

$$\frac{dy}{dt} + [p(t)y + q(t)]y + q(t)y = r(t), \tag{A.2}$$

which, upon defining the integrating factor

$$u(t) := u(0)\exp\left[\int_0^t p(x)y(x) + q(x)\right]dx , \tag{A.3}$$

becomes

$$\frac{d}{dt}(uy) = -q(t)uy + r(t)u . \tag{A.4}$$

On the other hand, from Eq. (A.3) we have

$$\frac{du}{dt} = p(t)uy + q(t)u . \tag{A.5}$$

Eqs. (A.4) and (A.5) together can be written as

$$\frac{d}{dt}\begin{bmatrix} uy \\ u \end{bmatrix} = \mathbf{A}\begin{bmatrix} uy \\ u \end{bmatrix}, \tag{A.6}$$

where

$$\mathbf{A} := \begin{bmatrix} -q(t) & r(t) \\ p(t) & q(t) \end{bmatrix}. \tag{A.7}$$

We thus obtained a representation of the Riccati differential equation in the projective space (uy, u) as a system of two first order linear differential equations.

It follows from Eqs. (A.5) and (A.4) a second order linear differential equation for u:

$$\ddot{u} + a(t)\dot{u} + b(t)u = 0, \tag{A.8}$$

where $\ddot{u} = d^2u/dt^2$, $\dot{u} = du/dt$, and

$$a(t) = \frac{-\dot{p}(t)}{p(t)}, \tag{A.9}$$

$$b(t) = \left[\frac{\dot{p}(t)}{p(t)} - q(t)\right]q(t) - p(t)r(t) - \dot{q}(t). \tag{A.10}$$

Appendix B

The Kronecker delta symbol is

$$d_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{B.1}$$

The permutation symbol is

$$e_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{if any two indices of } (i,j,k) \text{ are repeated.} \end{cases} \tag{B.2}$$

In terms of the Kronecker delta symbol we have

$$e_{ijk} = \begin{vmatrix} d_{i1} & d_{i2} & d_{i3} \\ d_{j1} & d_{j2} & d_{j3} \\ d_{k1} & d_{k2} & d_{k3} \end{vmatrix} = \begin{vmatrix} d_{i1} & d_{j1} & d_{k1} \\ d_{i2} & d_{j2} & d_{k2} \\ d_{i3} & d_{j3} & d_{k3} \end{vmatrix}. \tag{B.3}$$

Then it follows that

$$e_{ijk} e_{pqr} = \begin{vmatrix} d_{i1} & d_{i2} & d_{i3} \\ d_{j1} & d_{j2} & d_{j3} \\ d_{k1} & d_{k2} & d_{k3} \end{vmatrix} \begin{vmatrix} d_{p1} & d_{q1} & d_{r1} \\ d_{p2} & d_{q2} & d_{r2} \\ d_{p3} & d_{q3} & d_{r3} \end{vmatrix} = \begin{vmatrix} d_{ip} & d_{iq} & d_{ir} \\ d_{jp} & d_{jq} & d_{jr} \\ d_{kp} & d_{kq} & d_{kr} \end{vmatrix}. \tag{B.4}$$

If let $p = k$ in Eq. (B.4), we obtain the $e - d$ identity:

$$e_{ijk} e_{kqr} = d_{iq} d_{jr} - d_{ir} d_{jq} . \tag{B.5}$$

Define the inner product and cross product of $\mathbf{A}, \mathbf{B} \in R^3$ to be

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i , \tag{B.6}$$

$$\mathbf{A} \times \mathbf{B} = e_{ijk} A_j B_k \mathbf{e}_i , \tag{B.7}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are standard bases of R^3. Or write the latter in terms of the component of $\mathbf{A} \times \mathbf{B}$:

$$(\mathbf{A} \times \mathbf{B})_i = e_{ijk} A_j B_k . \tag{B.8}$$

By Eqs. (B.5)-(B.8) it is easy to prove the following results:

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$,

(b) $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$,

(c) $(\mathbf{A} \times \mathbf{B}) \cdot [(\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A})] = [\mathbf{A}, \mathbf{B}, \mathbf{C}]^2$.

Proof of (a):

$$
\begin{aligned}
&[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_{i} \\
&= e_{ijk}\,A_{j}\,(\mathbf{B}\times\mathbf{C})_{k} \\
&= e_{ijk}\,A_{j}\,e_{kmn}B_{m}C_{n} \\
&= (d_{im}d_{jn} - d_{in}d_{jm})A_{j}\,B_{m}C_{n} \\
&= d_{im}d_{jn}A_{j}B_{m}C_{n} - d_{in}d_{jm}A_{j}B_{m}C_{n} \\
&= A_{n}B_{i}C_{n} - A_{m}B_{m}C_{i} \\
&= (\mathbf{A}\cdot\mathbf{C})B_{i} - (\mathbf{A}\cdot\mathbf{B})C_{i}.
\end{aligned}
$$

Proof of (b):

$$
\begin{aligned}
&(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) \\
&= (\mathbf{A}\times\mathbf{B})_{i}\,(\mathbf{C}\times\mathbf{D})_{i} \\
&= e_{ijk}\,A_{j}B_{k}e_{imn}C_{m}D_{n} \\
&= (d_{jm}d_{kn} - d_{jn}d_{km})A_{j}B_{k}C_{m}D_{n} \\
&= d_{jm}d_{kn}A_{j}B_{k}C_{m}D_{n} - d_{jn}d_{km}A_{j}B_{k}C_{m}D_{n} \\
&= A_{m}B_{n}C_{m}D_{n} - A_{n}B_{m}C_{m}D_{n} \\
&= (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C}).
\end{aligned}
$$

Proof of(c):

$$
\begin{aligned}
&(\mathbf{A}\times\mathbf{B})\cdot[(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})] \\
&= (\mathbf{A}\times\mathbf{B})_{i}\,[(\mathbf{B}\times\mathbf{C})\times(\mathbf{C}\times\mathbf{A})]_{i} \\
&= e_{ijk}\,A_{j}B_{k}e_{imn}\,(\mathbf{B}\times\mathbf{C})_{m}\,(\mathbf{C}\times\mathbf{A})_{n} \\
&= (d_{jm}d_{kn} - d_{jn}d_{km})A_{j}B_{k}\,(\mathbf{B}\times\mathbf{C})_{m}(\mathbf{C}\times\mathbf{A})_{n} \\
&= d_{jm}d_{kn}A_{j}B_{k}(\mathbf{B}\times\mathbf{C})_{m}\,(\mathbf{C}\times\mathbf{A})_{n} - d_{jn}d_{km}A_{j}B_{k}(\mathbf{B}\times\mathbf{C})_{m}\,(\mathbf{C}\times\mathbf{A})_{n} \\
&= A_{m}B_{n}\,(\mathbf{B}\times\mathbf{C})_{m}\,(\mathbf{C}\times\mathbf{A})_{n} - A_{n}B_{m}(\mathbf{B}\times\mathbf{C})_{m}(\mathbf{C}\times\mathbf{A})_{n} \\
&= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) - \mathbf{A}\cdot(\mathbf{C}\times\mathbf{A})\mathbf{B}\cdot(\mathbf{B}\times\mathbf{C}) \\
&= [\mathbf{A},\mathbf{B},\mathbf{C}]^{2}.
\end{aligned}
$$

Appendix C

Define the following operations:

$$\nabla f = \frac{\partial f}{\partial x_i}\mathbf{e}_i = f_{,i}\mathbf{e}_i, \tag{C.1}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_i}{\partial x_i} = A_{i,i}, \tag{C.2}$$

$$\nabla \times \mathbf{A} = e_{ijk}\frac{\partial A_k}{\partial x_j}\mathbf{e}_i = e_{ijk}A_{k,j}\mathbf{e}_i. \tag{C.3}$$

By Eqs. (C.1)-(C.3) and (B.6) it is easy to prove the following results:

(a) $\nabla \cdot (\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$,

(b) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$,

(c) $\mathbf{A}\times(\nabla\times\mathbf{A}) = \nabla\|\mathbf{A}\|^2/2 - (\mathbf{A}\cdot\nabla)\mathbf{A}$,

(d) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B})$,

(e) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$.

Proof of(a):

$$\begin{aligned}
&\nabla\cdot(\mathbf{A}\times\mathbf{B}) \\
&= (\mathbf{A}\times\mathbf{B})_{i,i} = (e_{ijk}A_jB_k)_{,i} \\
&= e_{ijk}(A_{j,i}B_k + A_jB_{k,i}) \\
&= e_{ijk}A_{j,i}B_k + e_{ijk}A_jB_{k,i} \\
&= B_k e_{kij}A_{j,i} - A_j e_{jik}B_{k,i} \\
&= B_k(\nabla\times\mathbf{A})_k - A_j(\nabla\times\mathbf{B})_j \\
&= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}).
\end{aligned}$$

Proof of(b):

$$\begin{aligned}
&[\nabla\times(\nabla\times\mathbf{A})]_i \\
&= e_{ijk}(\nabla\times\mathbf{A})_{k,j} \\
&= e_{ijk}(e_{kmn}A_{n,m})_{,j} \\
&= e_{ijk}e_{kmn}A_{n,mj} \\
&= (d_{im}d_{jn} - d_{in}d_{jm})A_{n,mj} \\
&= d_{im}d_{jn}A_{n,mj} - d_{in}d_{jm}A_{n,mj} \\
&= A_{j,ij} - A_{i,jj} \\
&= [\nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}]_i.
\end{aligned}$$

Proof of(c):

$$
\begin{aligned}
&[\mathbf{A}\times(\nabla\times\mathbf{A})]_{\mathrm{i}} \\
&= e_{\mathrm{ijk}}\,\mathrm{A}_{\mathrm{j}}\,(\nabla\times\mathbf{A})_{\mathrm{k}} \\
&= e_{\mathrm{ijk}}\,\mathrm{A}_{\mathrm{j}}\,e_{\mathrm{kmn}}\mathrm{A}_{\mathrm{n,m}} \\
&= (d_{\mathrm{im}}d_{\mathrm{jn}} - d_{\mathrm{in}}d_{\mathrm{jm}})\mathrm{A}_{\mathrm{j}}\,\mathrm{A}_{\mathrm{n,m}} \\
&= d_{\mathrm{im}}d_{\mathrm{jn}}\mathrm{A}_{\mathrm{j}}\mathrm{A}_{\mathrm{n,m}} - d_{\mathrm{in}}d_{\mathrm{jm}}\mathrm{A}_{\mathrm{j}}\mathrm{A}_{\mathrm{n,m}} \\
&= \mathrm{A}_{\mathrm{n}}\mathrm{A}_{\mathrm{n,i}} - \mathrm{A}_{\mathrm{m}}\mathrm{A}_{\mathrm{i,m}} \\
&= \left[\nabla\|\mathbf{A}\|^2/2\right]_{\mathrm{i}} - [(\mathbf{A}\cdot\nabla)\mathbf{A}]_{\mathrm{i}}.
\end{aligned}
$$

Proof of(d):

$$
\begin{aligned}
&[\nabla\times(\mathbf{A}\times\mathbf{B})]_i \\
&= e_{ijk}(\mathbf{A}\times\mathbf{B})_{k,j} \\
&= e_{ijk}e_{kmn}(A_m B_n)_{,j} \\
&= (d_{im}d_{jn} - d_{in}d_{jm})(A_m B_{n,j} + A_{m,j}B_n) \\
&= d_{im}d_{jn}A_m B_{n,j} + d_{im}d_{jn}A_{m,j}B_n - d_{in}d_{jm}A_m B_{n,j} - d_{in}d_{jm}A_{m,j}B_n \\
&= A_i B_{j,j} + A_{i,j}B_j - A_j B_{i,j} - A_{j,j}B_i \\
&= [\mathbf{A}(\nabla\cdot\mathbf{B})]_i + [(\mathbf{B}\cdot\nabla)\mathbf{A}]_i - [(\mathbf{A}\cdot\nabla)\mathbf{B}]_i - [\mathbf{B}(\nabla\cdot\mathbf{A})]_i.
\end{aligned}
$$

Proof of(e): Let us first compute

$$
\begin{aligned}
&[\mathbf{A}\times(\nabla\times\mathbf{B})]_{\mathrm{i}} \\
&= e_{\mathrm{ijk}}\,\mathrm{A}_{\mathrm{j}}\,(\nabla\times\mathbf{B})_{\mathrm{k}} \\
&= e_{\mathrm{ijk}}\,\mathrm{A}_{\mathrm{j}}\,e_{\mathrm{kmn}}\mathrm{B}_{\mathrm{n,m}} \\
&= (d_{\mathrm{im}}d_{\mathrm{jn}} - d_{\mathrm{in}}d_{\mathrm{jm}})\mathrm{A}_{\mathrm{j}}\,\mathrm{B}_{\mathrm{n,m}} \\
&= d_{\mathrm{im}}d_{\mathrm{jn}}\mathrm{A}_{\mathrm{j}}\mathrm{B}_{\mathrm{n,m}} - d_{\mathrm{in}}d_{\mathrm{jm}}\mathrm{A}_{\mathrm{j}}\mathrm{B}_{\mathrm{n,m}} \\
&= \mathrm{A}_{\mathrm{n}}\mathrm{B}_{\mathrm{n,i}} - \mathrm{A}_{\mathrm{m}}\mathrm{B}_{\mathrm{i,m}} \\
&= \mathrm{A}_{\mathrm{n}}\mathrm{B}_{\mathrm{n,i}} - [(\mathbf{A}\cdot\nabla)\mathbf{B}]_{\mathrm{i}}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&[\mathbf{B}\times(\nabla\times\mathbf{A})]_{\mathrm{i}} \\
&= \mathrm{B}_{\mathrm{n}}\mathrm{A}_{\mathrm{n,i}} - [(\mathbf{B}\cdot\nabla)\mathbf{A}]_{\mathrm{i}}.
\end{aligned}
$$

Adding the above two equations and noting that

$$
\mathrm{A}_{\mathrm{n}}\mathrm{B}_{\mathrm{n,i}} + \mathrm{B}_{\mathrm{n}}\mathrm{A}_{\mathrm{n,i}} = [\nabla(\mathbf{A}\cdot\mathbf{B})]_{\mathrm{i}},
$$

we obtain

$$
\begin{aligned}
&[\mathbf{A}\times(\nabla\times\mathbf{B})]_i + [(\mathbf{A}\cdot\nabla)\mathbf{B}]_i \\
&+ [\mathbf{B}\times(\nabla\times\mathbf{A})]_i + [(\mathbf{B}\cdot\nabla)\mathbf{A}]_i \\
&= [\nabla(\mathbf{A}\cdot\mathbf{B})]_i \,.
\end{aligned}
$$

This ends the proof.