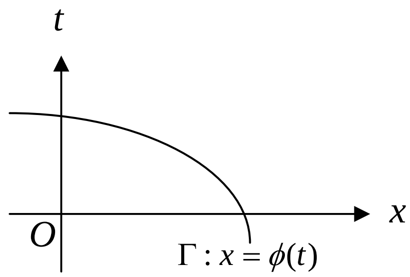
班級：______________學號：_______________姓名：_______________

海洋大學河海工程學系 2004 工程數學(四)期中考(Open Book)

1. 如果 $u = u(x,t)$ 是方程式 $u_t + a(u)u_x = b(u)$，在曲線 $\Gamma : x = \phi(t)$ 兩邊的解(即 $u(x,t)$ 在 Γ 兩邊各自 continuously differentiable, 且滿足上面方程式), 且 $u(x,t)$ 是 continuous



證明：$\phi(t)$ 滿足常微分方程式 $\dfrac{d\phi}{dt} = a(u(\phi(t),t))$ 其中 $a(u)$, $b(u)$ 是兩個給定的函數。 (20%)

$$\frac{\partial u}{\partial t}\frac{dt}{d\alpha} + \frac{\partial u}{\partial x}\frac{dx}{d\alpha} = \frac{\partial u}{\partial \alpha}$$

sol: $$\begin{cases} \dfrac{dt}{d\alpha} = 1, t(s,0) = s \\ \dfrac{dx}{d\alpha} = a(u), x(s,0) = \phi(s) \\ \dfrac{du}{d\alpha} = b(u), u(s,0) = u(\phi(s),s) \end{cases}$$ Since $u(x,y)$ 在 Γ 兩邊均連續可微

$\because \dfrac{dx}{dt} = a(u)$

會有 shock wave (Burger's equation)

解會不連續,因此除非 Γ 是特徵線,否則不可能 Γ 兩邊連續可微

Γ 滿足 $\dfrac{dx}{dt} = a(u), \dfrac{d\phi(t)}{dt} = a(u(\phi(t),t))$,得証之.

2. 考慮 Cauchy problem

$$\begin{cases} yu_x - xu_y = 0 \\ u(\cos\theta, \sin\theta) = g(\theta), \ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \\ u(0,y) = f(y), \ -1 \le y \le 1 \end{cases}$$

(1) 是否對所有 $f(y)$, $g(\theta)$ 皆有解 ? (10%)

(2) 如果 (1) 不對, $f(y)$ 及 $g(\theta)$ 要滿足什麼條件,上面問題才有解?又有多少解？是否對任何 $f(y)$ 及 $g(\theta)$, 此 Cauchy problem 皆無解? (10%)

sol: (1) $\begin{cases} \dfrac{dx}{dt} = y, x(s,0) = \cos s \\ \dfrac{dt}{dt} = -x, y(s,0) = \sin s \\ \dfrac{du}{dt} = 0, u(s,0) = g(s) \end{cases} \rightarrow \begin{cases} x(t,s) = \cos(t-s) \\ y(t,s) = -\sin(t-s) \\ u(t,s) = g(s) \end{cases}$

由此發現 Cauchy data 恰爲 characteristic line

可由 $\begin{vmatrix} -y & -x \\ -\sin s & \cos s \end{vmatrix} = \begin{vmatrix} \sin s & -\cos s \\ -\sin s & \cos s \end{vmatrix}$ =0 得知

因此,此 Cauchy data 無法推展出平面,而須用 Cauchy data 兩邊各自找到滿足 PDE 的解

(2) $\begin{cases} \dfrac{dx}{dy} = y, x(s,0) = 0 \\ \dfrac{dy}{dt} = -x, y(s,0) = s \\ \dfrac{du}{dt} = 0, u(s,0) = f(s) \end{cases} \Rightarrow u(s,t) = f(s)$

$$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow x^2 + y^2 = s^2$$

$$u(t,s) = f(\sqrt{x^2+y^2})$$

討論:若 $g(\theta) = f(1)$ 則有解,解爲 $u(x,y) = f(\sqrt{x^2+y^2})$ 若 $g(\theta) \neq f(1)$ 則無解

3. Given $u_x u_y = 1$, find the Monge cone at $(0,0,0)$. Try any method you can for $u_x u_y = 1$ subject to Cauchy data $u(s,s) = \sqrt{2}\, s$. (20%)

sol:(1) $F(x,y,z,p,q) \rightarrow pq - 1 = 0$ 令 p= λ ,q= $\dfrac{1}{\lambda}$

at (0,0,0)

$$\Rightarrow u - 0 = p(x-0) + q(y-0) \Rightarrow \begin{cases} u - 0 = \lambda(x-0) + \dfrac{1}{\lambda}(y-0) \\ 0 = 1(x-0) - \dfrac{1}{\lambda^2}(y-0) \end{cases} \Rightarrow \begin{cases} u = \lambda \bullet x + \dfrac{1}{\lambda} y \\ 0 = x - \dfrac{1}{\lambda^2} y \end{cases}$$

$$\begin{cases} u^2 = \lambda^2 x^2 + 2xy + \dfrac{1}{\lambda^2} y^2 \\ \dfrac{1}{\lambda^2} = \dfrac{x}{y} \end{cases} \Rightarrow u^2 = 4xy \text{ (Monge Cone)}$$

(2) $$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_u = 0 \\ F_p = q \\ F_q = p \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = q \\ \dfrac{dy}{dt} = p \\ \dfrac{du}{dt} = 2pq \\ \dfrac{dp}{dt} = 0 \\ \dfrac{dq}{dt} = 0 \end{cases} \quad \because \begin{cases} pq = 1 \\ p + q = \sqrt{2} \end{cases}$$

$$\begin{cases} p = \dfrac{\sqrt{2}+\sqrt{2}i}{2} \, or \, \dfrac{\sqrt{2}-\sqrt{2}i}{2} \\ q = \dfrac{\sqrt{2}-\sqrt{2}i}{2} \, or \, \dfrac{\sqrt{2}+\sqrt{2}i}{2} \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\sqrt{2}-\sqrt{2}i}{2} \, or \, \dfrac{\sqrt{2}+\sqrt{2}i}{2}, x(s,0) = s \\ \dfrac{dy}{dt} = \dfrac{\sqrt{2}+\sqrt{2}i}{2} \, or \, \dfrac{\sqrt{2}-\sqrt{2}i}{2}, y(s,0) = s \Rightarrow u = 2t + \sqrt{2}s \\ \dfrac{du}{dt} = 2, u(s,0) = \sqrt{2}s \end{cases}$$

$$\therefore \begin{cases} u(x,y) = \dfrac{\sqrt{2}+\sqrt{2}i}{2} x + \dfrac{\sqrt{2}-\sqrt{2}i}{2} y \\ u(x,y) = \dfrac{\sqrt{2}-\sqrt{2}i}{2} x + \dfrac{\sqrt{2}+\sqrt{2}i}{2} y \end{cases}$$

4. Solve $F(x, y, u, p, q) = p^2 + q + u = 0$ subject to the Cauchy data $u(s, 0) = s$. (20%)

$$\begin{cases} F_x = 0 \\ F_y = 0 \\ F_u = 1 \\ F_p = 2p \\ F_q = 1 \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = 2p \\ \dfrac{dy}{dt} = 1 \\ \dfrac{du}{dt} = 2p^2 + q \\ \dfrac{dp}{dt} = -p \\ \dfrac{dq}{dt} = -q \end{cases} \Rightarrow \begin{cases} x(0,s) = s \\ y(0,s) = 0 \\ u(0,s) = s \\ p(0,s) = 1 \\ q(0,s) = 1 - s \end{cases}$$

$p^2+q+u=0$

$p=1$

$q=-1-u=-1-s$

$\frac{dx}{dt}=2e^{-t}$, $x(0,s)=s$, $x=-2e^{-t}+s+2$

$\frac{du}{dt}=2e^{-t}-(1+s)e^{-t}$, $u(0,s)=s$, $u=-e^{-t}+(1+s)e^{-t}$

$x(t,s)=-2e^{-t}+s+2$

$y(t,s)=t$

$u(t,s)=-e^{-2t}+(1+s)e^{-t}$

$\therefore u(x,y)=e^{-2y}+(x-1)e^{-y}$

5. Solve $xu_x+yu_y=1+y^2$ subject to the $u(x,1)=1+x$. (20%)

sol: $\frac{du}{dt}=\frac{du}{dx}\frac{dx}{dt}+\frac{du}{dy}\frac{dy}{dt}$

$$\begin{cases}\frac{dx(t,s)}{dt}=x, x(s,0)=s\\ \frac{dy(t,s)}{dt}=y, y(s,0)=1\\ \frac{du(t,s)}{dt}=1+y^2, u(s,0)=1+s\end{cases} \Rightarrow \begin{cases}x(t,s)=c_1e^t\\ y(t,s)=c_2e^t\\ u(t,s)=t+\frac{1}{2}e^{2t}+s+\frac{1}{2}\end{cases} \Rightarrow \begin{cases}x(t,s)=se^t\\ y(t,s)=e^t\end{cases}$$

$\frac{du}{dt}=1+e^{2t}+c_3$, $u=t+\frac{1}{2}+e^{2t}+c_3$, $u(0,s)=\frac{1}{2}+c_3=1+s$, $c_3=\frac{1}{2}+s$

$u(t,s)=t+\frac{1}{2}e^{2t}+s+\frac{1}{2}$

$x=se^t$, $x=sy$, $s=\frac{x}{y}$

$y=e^t$

$y^2=e^{2t}$, $\ln y=t$

$\therefore \ u(x,y)=\frac{1}{2}y^2+\frac{x}{y}+\ln y+\frac{1}{2}$

6. 考慮方程式 $u^2(1+u_x^2+u_y^2)=1$, $u=u(x,y)$ 。

(a) 求它的 field of Monge cones. (10%)

(b) 求一解滿足下面 Cauchy data: $x=s, y=s, z=\frac{1}{2}s$ 。

問： s 要滿足什麼條件才使 $(s, s, \frac{1}{2}s)$ 是 non-characteristic. (10%)