海洋大學河海工程學系 2005 工程數學(四)第三次作業小考解答

1. Solve $u(x,y)$.

$$\begin{cases} u - xu_x - 0.5u_y^2 + x^2 = 0 \\ u(s,0) = s^2 - \dfrac{1}{6}s^4 \ \ Cauchy\ data \end{cases}$$

$$\Rightarrow F(x,y,u,p,q) = u - xp - 0.5q^2 + x^2 = 0\,, \ \Rightarrow \begin{cases} F_x = -p + 2x \\ F_y = 0 \\ F_u = 1 \\ F_p = -x \\ F_q = -q \end{cases}$$

(1)

$$\therefore \begin{cases} u(0,s) - x(0,s)\,p(0,s) - 0.5[q(0,s)]^2 + [x(0,s)]^2 = 0 \\ p(0,s) + 0 = 2s - \dfrac{2}{3}s^3 \end{cases} \qquad \Rightarrow \begin{cases} p(0,s) = 2s - \dfrac{2}{3}s^3 \\ q(0,s) = s^2 \ \ or \ \ -s^2 \end{cases}$$

(2)

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = F_p = -x,\ x(0,s) = s \\ \dfrac{dp}{dt} = -(F_x + F_u p) = -(-p + 2x + p) = -2x,\ p(0,s) = 2s - \dfrac{2}{3}s^3 \\ \dfrac{dq}{dt} = -(F_y + F_u q) = -q,\ q(0,s) = s^2 \ \ or \ \ -s^2 \\ \dfrac{dy}{dt} = F_q = -q,\ y(0,s) = 0 \\ \dfrac{du}{dt} = F_p p + F_q q = -2x - q^2,\ u(0,s) = s^2 - \dfrac{1}{6}s^4 \end{cases}$$

$$\Rightarrow \begin{cases} x(t,s) = se^{-t} \\ p(t,s) = 2se^{-t} - \dfrac{2}{3}s^3 \\ q(t,s) = s^2e^{-t} \ \ or \ \ -s^2e^{-t} \\ y(t,s) = s^2e^{-t} - s^2 \ \ or \ \ -(s^2e^{-t} - s^2) \\ u(t,s) = s^2e^{-2t} - \dfrac{2}{3}s^4e^{-t} + \dfrac{1}{2}s^4e^{-2t} \end{cases}$$

(3)

$$When \ \begin{cases} x(t,s) = se^{-t} \\ y(t,s) = s^2e^{-t} - s^2 \end{cases} \Rightarrow \begin{cases} e^{-t} = \dfrac{x}{s} \\ y = sx - s^2 \end{cases} \Rightarrow s = \frac{x \pm \sqrt{x^2 - 4y}}{2}$$

$$\therefore u(t,s) = s^2e^{-2t} - \frac{2}{3}s^4e^{-t} + \frac{1}{2}s^4e^{-2t}$$

$$u(x,y) = x^2 - \frac{2}{3}s^3x + \frac{1}{2}s^2x^2 = x^2 - \frac{2}{3}s(sx - y)x + \frac{1}{2}s^2x^2 = x^2 + \frac{2}{3}sxy - \frac{1}{6}s^2x$$

$$= x^2 - \frac{1}{12}x^4 + \frac{1}{2}x^2y + \left(\frac{xy}{3} - \frac{x^3}{12}\right)\sqrt{x^2 - 4y}$$

$u(x,y)=x^2-\frac{1}{12}x^4+\frac{1}{2}x^2y-(\frac{xy}{3}-\frac{x^3}{12})\sqrt{x^2-4y}$ is unmet.

Simile, if $x(t,s)=se^{-t}$ and $y(t,s)=-(s^2e^{-t}-s^2)$, than

$$u(x,y)=x^2-\frac{1}{12}x^4+\frac{1}{2}x^2y+(\frac{xy}{3}-\frac{x^3}{12})\sqrt{x^2+4y}$$

2. Solve $u(x,y)$.

$$\begin{cases} u_x^2+u_y^2=1 \\ u(s^2,s)=s \quad \textit{Cauchy data} \end{cases}$$

$\Rightarrow F(x,y,u,p,q)=p^2+q^2-1=0 \quad \Rightarrow F_x=0,\ F_y=0,\ F_u=0,\ F_p=2p,\ F_q=2q$

(1)

$$\begin{cases} p^2+q^2-1=0 \\ 2sp+q=1 \end{cases} \quad \Rightarrow \begin{cases} p(0,s)=0 \\ q(0,s)=1 \end{cases} \ or \ \begin{cases} p(0,s)=\frac{4s}{1+4s^2} \\ q(0,s)=\frac{1-4s^2}{1+4s^2} \end{cases}$$

(2)

$$\Rightarrow \begin{cases} \frac{dp}{dt}=-(F_x+F_up)=0,\ p(0,s)=0 \ or \ \frac{4s}{1+4s^2} & \Rightarrow p(t,s)=0 \ or \ \frac{4s}{1+4s^2} \\ \frac{dq}{dt}=-(F_y+F_uq)=0,\ q(0,s)=1 \ or \ \frac{1-4s^2}{1+4s^2} & \Rightarrow q(t,s)=1 \ or \ \frac{1-4s^2}{1+4s^2} \\ \frac{dx}{dt}=F_p=2p,\ x(0,s)=s^2 & \Rightarrow x(t,s)=s^2 \ or \ \frac{8s}{1+4s^2}t+s^2 \\ \frac{dy}{dt}=F_q=2q,\ y(0,s)=s & \Rightarrow y(t,s)=2t+s \ or \ \frac{2-8s^2}{1+4s^2}t+s \\ \frac{du}{dt}=F_pp+F_qq=2(p^2+q^2)=2,\ u(0,s)=s & \Rightarrow u(t,s)=2t+s \end{cases}$$

(3)

$\therefore u(x,y)=y,\ \ when\ \ x(t,s)=s^2\ \ and\ \ y(t,s)=2t+s$ or

$u(t,s)=2t+s$, $\ when\ \ x(t,s)=\frac{8s}{1+4s^2}t+s^2\ \ and\ \ y(t,s)=\frac{2-8s^2}{1+4s^2}t+s$.