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Inverses of 2×2 Block Matrices

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Abstract—In this paper, the authors give explicit inverse formulae for 2×2 block matrices with three different partitions. Then these results are applied to obtain inverses of block triangular matrices and various structured matrices such as Hamiltonian, per-Hermitian, and centro-Hermitian matrices.

Keywords—2×2 block matrix, Inverse matrix, Structured matrix.

1. INTRODUCTION

This paper is devoted to the inverses of 2×2 block matrices. First, we give explicit inverse formulae for a 2×2 block matrix

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1}, \tag{1.1}$$

with three different partitions. Then these results are applied to obtain inverses of block triangular matrices and various structured matrices such as bisymmetric, Hamiltonian, per-Hermitian, and centro-Hermitian matrices. In the end, we briefly discuss the completion problems of a 2×2 block matrix and its inverse, which generalizes this problem.

The inverse formula (1.1) of a 2×2 block matrix appears frequently in many subjects and has long been studied. Its inverse in terms of A^{-1} or D^{-1} can be found in standard textbooks on linear algebra, e.g., [1–3]. However, we give a complete treatment here. Some papers, e.g., [4,5], deal with its inverse in terms of the generalized inverse of A. Needless to say, a lot of research is devoted to the generalized inverse of the 2×2 block matrix, e.g., [6–8]. But this paper is not in this direction.

There are many related papers on the 2×2 block matrix. The Schur complement $D - CA^{-1}B$ of A in (1.1) has been studied by several mathematicians, e.g., [9–11]. Lazutkin [12] studies the signature of a symmetric 2×2 block matrix. Bapat and Kwong [13] obtain an inequality for the Schur product of positive definite 2×2 block matrices.

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This paper has three objectives. First, we completely list all the relevant formulae for a 2×2 block matrix and its inverse. Although it is nothing but a mechanical exercise, some of the results do not appear in the literature. Second, we explore matrices with symmetric structures related to a 2×2 block matrix.

We present this paper in the traditional way, though our formulae can also be proved using computer algebra systems. In fact, the package NCAlgebra [14] for noncommutative algebra has been used to solve certain 2×2 and 3×3 block matrix completion problems. The methodology of the solution procedure is explained in [15]. So our final objective is to indicate this fact and encourage further study of these techniques.

Only complex matrices will be considered in this paper, but most of our results can be extended to matrices with elements in an arbitrary field. This paper is organized as follows. In Section 2, we derive several formulae for the inverse of a 2×2 block matrix with three different partitions. In Section 3, we apply these results to get the inverses of 2×2 block triangular matrices. In Section 4, we apply our formulae to matrices with certain structures. In the last section, we indicate the related completion problems of a 2×2 block matrix and its inverse, and the possible computer theorem proving of matrix theory.

2. INVERSE FORMULAE

A nonsingular square matrix R and its inverse R^{-1} can be partitioned into 2×2 blocks as

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad R^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}. \tag{2.1}$$

To make the multiplication of R by R^{-1} and R^{-1} by R possible, the sizes of all blocks cannot be arbitrary. Assume A, B, C, and D have sizes $k \times m$, $k \times n$, $l \times m$, and $l \times n$, respectively, with $k + l = m + n$; then the sizes of E, F, G, and H must be $m \times k$, $m \times l$, $n \times k$, and $n \times l$, respectively. In other words, R^{-1} is in the transposed partition of R.

In this section, we shall write down the formulae for E, F, G, and H in terms of A, B, C, and D. We assume one of the blocks A, B, C, or D is a nonsingular square matrix to avoid generalized inverses. Thus, we have only three possible partitions:

- square diagonal partition: $k = m$ and $l = n$,
- square off-diagonal partition: $k = n$ and $l = m$,
- all-square partition: $k = l = m = n$.

The original matrix R and its inverse R^{-1}, of course, must have even dimension in the all-square partition.

First, we consider the square diagonal partition of R and R^{-1}. In this case, A, D, E, H are square matrices, A and E have the same size, and so do D and H. The following theorem is well known and can be found in [3, Problem 1.6.7].

THEOREM 2.1.

(i) *Assume A is nonsingular; then the matrix R in (2.1) is invertible if and only if the Schur complement $D - CA^{-1}B$ of A is invertible, and*

$$R^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B\left(D - CA^{-1}B\right)^{-1}CA^{-1} & -A^{-1}B\left(D - CA^{-1}B\right)^{-1} \\ -\left(D - CA^{-1}B\right)^{-1}CA^{-1} & \left(D - CA^{-1}B\right)^{-1} \end{bmatrix}. \tag{2.2}$$

(ii) *Assume D is nonsingular; then the matrix R is invertible if and only if the Schur complement $A - BD^{-1}C$ of D is invertible, and*

$$R^{-1} = \begin{bmatrix} \left(A - BD^{-1}C\right)^{-1} & -\left(A - BD^{-1}C\right)^{-1}BD^{-1} \\ -D^{-1}C\left(A - BD^{-1}C\right)^{-1} & D^{-1} + D^{-1}C\left(A - BD^{-1}C\right)^{-1}BD^{-1} \end{bmatrix}. \tag{2.3}$$

It is clear that these two set of formulae are used in different situations, and they are equivalent if both A and D are nonsingular. References [1, Theorem 8.2.1] and [2, 0.7.3] give mixed expressions for R^{-1} by combining these two sets of formulae.

There are several ways to prove this theorem, but it is rather trivial in view of the block Gaussian elimination of R [16, Exercise 2.6.15]. More precisely,

$$R = \begin{bmatrix} I & O \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & O \\ O & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & I \end{bmatrix},$$

for the first part of Theorem 2.1, and

$$R = \begin{bmatrix} I & BD^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & O \\ O & D \end{bmatrix} \begin{bmatrix} I & O \\ D^{-1}C & I \end{bmatrix},$$

for the second part.

On the other hand, consider the square off-diagonal partition of R and R^{-1}. In this case B, C, F, G are square matrices, B and G have the same size, and so do C and F. With suitable permutation of rows and columns, we can transform R and R^{-1} to the square diagonal partition. For example, let J be the matrix with 1 in the secondary diagonal and 0 elsewhere. Then RJ reverses the order of columns of R and becomes the square diagonal partition. Hence, we can use Theorem 2.1 on

$$RJ = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} O & J \\ J & O \end{bmatrix} = \begin{bmatrix} BJ & AJ \\ DJ & CJ \end{bmatrix}$$

to get

$$R^{-1} = J(RJ)^{-1}.$$

But for the completeness and later use, we include their formulae here.

THEOREM 2.2.

(i) *Assume B is nonsingular; then the matrix R in (2.1) is invertible if and only if the Schur complement $C - DB^{-1}A$ of B is invertible and*

$$R^{-1} = \begin{bmatrix} -\left(C - DB^{-1}A\right)^{-1} DB^{-1} & \left(C - DB^{-1}A\right)^{-1} \\ B^{-1} + B^{-1}A\left(C - DB^{-1}A\right)^{-1} DB^{-1} & -B^{-1}A\left(C - DB^{-1}A\right)^{-1} \end{bmatrix}. \tag{2.4}$$

(ii) *Assume C is nonsingular; then the matrix R is invertible if and only if the Schur complement $B - AC^{-1}D$ of C is invertible and*

$$R^{-1} = \begin{bmatrix} -C^{-1}D\left(B - AC^{-1}D\right)^{-1} & C^{-1} + C^{-1}D\left(B - AC^{-1}D\right)^{-1} AC^{-1} \\ \left(B - AC^{-1}D\right)^{-1} & -\left(B - AC^{-1}D\right)^{-1} AC^{-1} \end{bmatrix}. \tag{2.5}$$

Similarly, these two sets of formulae are used in different situations, and they are equivalent if both B and C are nonsingular.

Finally, we consider the all-square partition; i.e., all blocks in R and R^{-1} are square. In this case, R and R^{-1} must be of even size. Since this partition can be regarded as the square diagonal partition and the square off-diagonal partition, the previous two theorems are applicable, but used under different assumptions. In some special cases, these formulae are identical:

- (2.2) and (2.4) are equivalent if A and B are invertible;
- (2.2) and (2.5) are equivalent if A and C are invertible;
- (2.3) and (2.4) are equivalent if B and D are invertible;
- (2.3) and (2.4) are equivalent if C and D are invertible.

We remark that for a nonsingular matrix R, it is possible that all its blocks A, B, C, and D are singular. So all the above formulae fail to compute R^{-1}. A typical example is the all-square partition of permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In fact, all of its square diagonal partitions fail to have nonsingular A or D. However, the square off-diagonal partition with $m = 1$ and $n = 3$ can be applied to get its inverse. Therefore, different partitions give us more choices to find the desired inverse when other methods break down, and each one has its own value as we shall see in the next two sections.

3. BLOCK TRIANGULAR MATRICES

In this section, we apply our main theorems to 2×2 block diagonal and block triangular matrices. First, we have the trivial consequence for the inverses of block diagonal and block secondary diagonal matrices.

COROLLARY 3.1.

(i) *For the square diagonal partition,* $\left[\begin{smallmatrix} A & O \\ O & D \end{smallmatrix}\right]$ *is invertible if and only if* A *and* D *are invertible, and its inverse is* $\left[\begin{smallmatrix} A^{-1} & O \\ O & D^{-1} \end{smallmatrix}\right]$.

(ii) *For the square off-diagonal partition,* $\left[\begin{smallmatrix} O & B \\ C & O \end{smallmatrix}\right]$ *is invertible if and only if* B *and* C *are invertible, and its inverse is* $\left[\begin{smallmatrix} O & C^{-1} \\ B^{-1} & O \end{smallmatrix}\right]$.

Notice that the inverse of a block diagonal matrix is also block diagonal. Similarly, the inverse of a block secondary diagonal matrix is block secondary diagonal too, but in transposed partition so that there is a switch between B and C. This corollary is also easy to extend to $n \times n$ block diagonal and secondary diagonal matrices.

In the rest of this section, we will study the inverses of block triangular matrices. By Theorems 2.1 and 2.2, we have the following corollary.

COROLLARY 3.2. *Consider block upper triangular matrix* $\left[\begin{smallmatrix} A & B \\ O & D \end{smallmatrix}\right]$.

(i) *For the square diagonal partition, it is invertible if and only if* A *and* D *are invertible, and it has inverse*

$$\begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ O & D^{-1} \end{bmatrix}. \tag{3.1}$$

(ii) *For the square off-diagonal partition with* B *nonsingular, it is invertible if and only if* $DB^{-1}A$ *is invertible, and it has inverse*

$$\begin{bmatrix} \left(DB^{-1}A\right)^{-1} DB^{-1} & -\left(DB^{-1}A\right)^{-1} \\ B^{-1} - B^{-1}A\left(DB^{-1}A\right)^{-1} DB^{-1} & B^{-1}A\left(DB^{-1}A\right)^{-1} \end{bmatrix}. \tag{3.2}$$

Clearly, the inverse of a block upper triangular matrix is block upper triangular only in the square diagonal partition. In general this is not true for the square off-diagonal partition. Moreover, if the partition is in fact an all-square partition and A, B, and D are all invertible, then (3.2) is equivalent to (3.1).

Similarly, for the block lower triangular matrix in the square diagonal partition,

$$\begin{bmatrix} A & O \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & O \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}.$$

For the square off-diagonal partition,

$$\begin{bmatrix} A & O \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} C^{-1}D\left(AC^{-1}D\right)^{-1} & C^{-1} - C^{-1}D\left(AC^{-1}D\right)^{-1}AC^{-1} \\ -\left(AC^{-1}D\right)^{-1} & \left(AC^{-1}D\right)^{-1}AC^{-1} \end{bmatrix}.$$

There are two more possibilities, namely, $A = O$ or $D = O$. For the former case, Theorems 2.1 and 2.2 are reduced to the following corollary.

COROLLARY 3.3. *Consider matrix* $\begin{bmatrix} O & B \\ C & D \end{bmatrix}$.

(i) *For the square off-diagonal partition, it is invertible if and only if B and C are also invertible, and it has inverse*

$$\begin{bmatrix} -C^{-1}DB^{-1} & C^{-1} \\ B^{-1} & O \end{bmatrix}. \tag{3.3}$$

(ii) *For the square diagonal partition with D nonsingular, it is invertible if and only if $BD^{-1}C$ is also invertible, and it has inverse*

$$\begin{bmatrix} -\left(BD^{-1}C\right)^{-1} & \left(BD^{-1}C\right)^{-1}BD^{-1} \\ D^{-1}C\left(BD^{-1}C\right)^{-1} & D^{-1} - D^{-1}C\left(BD^{-1}C\right)^{-1}BD^{-1} \end{bmatrix}. \tag{3.4}$$

In the first part of the theorem, its inverse is still sparse but in the transposed position. In the second half, the sparsity will be destroyed in general. Moreover, if the partition is in fact an all-square partition and B, C, D are all invertible, then (3.4) is equivalent to (3.3).

Similar results hold for the square off-diagonal partition

$$\begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} O & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix},$$

and for the square diagonal partition

$$\begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} - A^{-1}B\left(CA^{-1}B\right)^{-1}CA^{-1} & A^{-1}B\left(CA^{-1}B\right)^{-1} \\ \left(CA^{-1}B\right)^{-1}CA^{-1} & -\left(CA^{-1}B\right)^{-1} \end{bmatrix}.$$

4. STRUCTURED MATRICES

In this section, we will apply our main theorems to structured matrices, which includes bisymmetric, Hamiltonian, Hankel, Toeplitz, circulant, Hermitian, per-Hermitian, centro-Hermitian, and their skew Hermitian matrices.

The natural partition for a Hermitian or symmetric matrix is the square diagonal partition, which preserves the symmetry of the diagonal blocks. On the contrary, the square off-diagonal partition will, in general, spoil the symmetry of Hermitian matrices. However, Theorem 2.1 or Theorem 2.2 is still applicable for a Hermitian matrix of even size in the all-square partition. In summary, we have the following corollary.

COROLLARY 4.1.

(i) *A matrix is Hermitian if and only if it has the form* $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ *in the square diagonal partition, where A and D are Hermitian. Its inverse can be computed by*

$$\begin{bmatrix} A^{-1} + A^{-1}B\left(D - B^*A^{-1}B\right)^{-1}B^*A^{-1} & -A^{-1}B\left(D - B^*A^{-1}B\right)^{-1} \\ -\left(D - B^*A^{-1}B\right)^{-1}B^*A^{-1} & \left(D - B^*A^{-1}B\right)^{-1} \end{bmatrix} \tag{4.1}$$

or

$$\begin{bmatrix} \left(A - BD^{-1}B^*\right)^{-1} & -\left(A - BD^{-1}B^*\right)^{-1}BD^{-1} \\ -D^{-1}B^*\left(A - BD^{-1}B^*\right)^{-1} & D^{-1} + D^{-1}B^*\left(A - BD^{-1}B^*\right)^{-1}BD^{-1} \end{bmatrix}, \tag{4.2}$$

if the required inverses exist.

(ii) *The Hermitian matrix* $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ *in the all-square partition has inverse*

$$\begin{bmatrix} -\left(B^* - DB^{-1}A\right)^{-1} DB^{-1} & \left(B^* - DB^{-1}A\right)^{-1} \\ B^{-1} + B^{-1}A\left(B^* - DB^{-1}A\right)^{-1} DB^{-1} & -B^{-1}A\left(B^* - DB^{-1}A\right)^{-1} \end{bmatrix} \tag{4.3}$$

$$= \begin{bmatrix} -B^{-*}D\left(B - AB^{-*}D\right)^{-1} & B^{-*} + B^{-*}D\left(B - AB^{-*}D\right)^{-1} AB^{-*} \\ \left(B - AB^{-*}D\right)^{-1} & -\left(B - AB^{-*}D\right)^{-1} AB^{-*} \end{bmatrix}, \tag{4.4}$$

if the required inverses exist.

Here we use the notation $B^{-*} = (B^{-1})^* = (B^*)^{-1}$. These results come directly from Theorems 2.1 and 2.2. Notice that the inverses

$$A^{-1}, \qquad \left(D - B^*A^{-1}B\right)^{-1}, \qquad D^{-1}, \qquad \text{and} \qquad \left(A - BD^{-1}B^*\right)^{-1}$$

in Part (i) can also be calculated by the same partition method. Thus, we have a recursive method to obtain the inverses of Hermitian matrices, which is useful in practical and parallel computing.

It is well known that the inverse $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ of a Hermitian matrix is also Hermitian. Hence, for the square diagonal partition, E and H are Hermitian, and $G = F^*$. This fact can also be checked easily from the above inverse formulae. So it suffices to compute only one of G and F, and half of the entries of E and H.

As a special case, we consider the positive definite matrix, which is Hermitian automatically [2, p. 397]. Let it be partitioned as $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$; then its principal submatrices A and D are positive definite too. So both A and D are nonsingular. The inverse of such a matrix can be computed using the same formulae (4.1)–(4.4). Notice that its inverse matrix is positive definite and so are the principal submatrices of this inverse. Therefore, all the diagonal blocks in (4.1)–(4.4) are all positive definite. Horn and Johnson [2, p. 472] point out the same fact, but only on $(A - BD^{-1}B^*)^{-1}$ and $(D - B^*A^{-1}B)^{-1}$.

Now we turn to the skew-Hermitian or skew-symmetric matrix. The square diagonal partition is the right choice in order to preserve skew-symmetry. In fact, we have the following corollary.

COROLLARY 4.2. *A matrix is skew-Hermitian if and only if it has the representation* $\begin{bmatrix} A & B \\ -B^* & D \end{bmatrix}$ *in the square diagonal partition, with A and D skew-Hermitian. Its inverse can be computed by (2.2) and (2.3). For the all-square partition, in addition (2.4) and (2.5) can be used.*

It is easy to see that the derived inverse matrix is skew-Hermitian too. We remark that it suffices to consider the inverse of a skew-symmetric matrix of even order, since a skew-symmetric matrix of odd order must be singular and has no inverse.

A bisymmetric matrix is a real matrix of the form

$$\begin{bmatrix} A & B \\ -B^\top & D \end{bmatrix}, \tag{4.5}$$

such that its diagonal blocks A and D are symmetric negative semidefinite of the same size, and the remaining matrix $\begin{bmatrix} O & B \\ -B^\top & O \end{bmatrix}$ is skew-symmetric. It is straightforward to show that a bisymmetric matrix is negative semidefinite as well. Such matrices occur in the linear complementarity problems of quadratic programming; for example, see [17].

Since a bisymmetric matrix is in the all-square partition, we can get its inverse $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ by Theorem 2.1 as well as Theorem 2.2.

COROLLARY 4.3. *The inverse of the bisymmetric matrix (4.5) can be computed according to (2.2)–(2.5).*

Notice that equations (2.2) and (2.3) hold even though A and D have different sizes. The inverse of a bisymmetric matrix is bisymmetric too. To see this, we first check that $G = -F^\top$, E

and H are symmetric. Since this bisymmetric matrix has an inverse, it must be negative definite, and so must its inverse and corresponding principal submatrices E and H.

A matrix $R = (r_{ij})$ is called per-Hermitian if $r_{ij} = \bar{r}_{n+1-j,n+1-i}$ for all i and j [18]. In short, $R = JR^*J$, where J is the matrix defined in Section 2. A real per-Hermitian matrix is called persymmetric, or secondary symmetric in [19,20], which elements are symmetric with respect to its secondary diagonal. The natural partition for a per-Hermitian matrix is the square off-diagonal partition, which preserves the symmetry of the off-diagonal blocks. On the contrary, the square diagonal partition, except the all-square one, will spoil the symmetry of the per-Hermitian matrix in general.

COROLLARY 4.4. *A matrix is per-Hermitian if and only if it has the form* $\begin{bmatrix} A & B \\ C & JA^*J \end{bmatrix}$ *in the square off-diagonal partition, where B and C are per-Hermitian. Its inverse can be computed by using (2.4) and (2.5). In particular, if its partition is the all-square one, then in addition both (2.2) and (2.3) can be used.*

Similarly, the inverses

$$B^{-1}, \qquad \left(C - JA^*JB^{-1}A\right)^{-1}, \qquad C^{-1} \qquad \text{and} \qquad \left(B - AC^{-1}JA^*J\right)^{-1}$$

in (2.4) and (2.5) can be calculated recursively by the same partition method.

It is trivial to see that the inverse $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ of a per-Hermitian matrix R is also per-Hermitian, since

$$R^{-1} = (JR^*J)^{-1} = J\left(R^{-1}\right)^* J.$$

Hence, for the square off-diagonal partition, F and G are per-Hermitian, and $H = JE^*J$. This fact can also be checked easily from all its inverse formulae. So it suffices to compute only one of E and H, and half of the entries of F and G.

Similarly, a matrix R is called skew-per-Hermitian if $R = -JR^*J$ [18]. Like per-Hermitian matrices, the square off-diagonal partition is the right choice in order to preserve skew-persymmetry of such matrices. It is also easy to see that the derived inverse matrix is skew-per-Hermitian too.

COROLLARY 4.5. *A matrix is skew-per-Hermitian if and only if it has the form* $\begin{bmatrix} A & B \\ C & -JA^*J \end{bmatrix}$ *in the square off-diagonal partition, with B and C skew-per-Hermitian. Its inverse can be computed by (2.4) and (2.5). For such a matrix in the all-square partition, (2.2) and (2.3) can also be used.*

A real skew-per-Hermitian matrix is usually called skew-persymmetric matrix, or secondary skew-symmetric in [19,20]. We remark that every skew-persymmetric matrix of odd order is singular and has no inverse. This can be easily verified as follows. We first notice that the determinant of the matrix J is either 1 or -1. Then an $n \times n$ skew-persymmetric matrix R satisfies

$$\det R = (-1)^n \det J \cdot \det R^\top \cdot \det J = (-1)^n \det R.$$

Hence, $\det R$ vanishes when n is odd.

A Hamiltonian matrix is a matrix of the form

$$\begin{bmatrix} A & B \\ C & -A^\top \end{bmatrix}, \tag{4.6}$$

where B and C are symmetric of the same size. Such a matrix is related to the algebraic Riccati equation in control theory [21]. Since it is in the all-square partition, both Theorems 2.1 and 2.2 are applicable.

COROLLARY 4.6. *The inverse of the Hamiltonian matrix (4.6) can be computed by (2.2)–(2.5).*

Notice that (2.4) and (2.5) hold even when B and C are of different sizes. Let its inverse be $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$. It is easy to show, from our inverse formulae, that F and G are symmetric as well as $H = -E^\top$. Hence, the inverse of a Hamiltonian matrix is also Hamiltonian.

A matrix $R = (r_{ij})$ is called centro-Hermitian if $r_{ij} = \bar{r}_{n+1-i,n+1-j}$ for all i and j [22,23]. In other words, $R = J\bar{R}J$. In fact, we have the following corollary.

COROLLARY 4.7. *A matrix is centro-Hermitian of even order if and only if it has the form* $\begin{bmatrix} A & BJ \\ J\bar{B} & J\bar{A}J \end{bmatrix}$, *where A and B have the same size. Since it corresponds to the all-square partition, all formulae (2.2)–(2.5) can be used to compute its inverse.*

Since

$$R^{-1} = J^{-1}\bar{R}^{-1}J^{-1} = J\overline{R^{-1}}J,$$

the inverse $\begin{bmatrix} E & F \\ G & H \end{bmatrix}$ of a centro-Hermitian matrix is also centro-Hermitian. So it suffices to compute only one of E and H, and only one of F and G. A real centro-Hermitian matrix is called centrosymmetric, which entries are symmetric with respect to the center of the matrix. Every centrosymmetric matrix of even order is similar to a block diagonal matrix [24,25], i.e.,

$$\begin{bmatrix} A & BJ \\ JB & JAJ \end{bmatrix} = K^{-1}\begin{bmatrix} A+B & O \\ O & A-B \end{bmatrix} K,$$

where

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix} I & J \\ -I & J \end{bmatrix}. \tag{4.7}$$

In addition to Corollary 4.7, its inverse, if it exists, can be computed by

$$K^{-1}\begin{bmatrix} A+B & O \\ O & A-B \end{bmatrix}^{-1} K = K^{-1}\begin{bmatrix} (A+B)^{-1} & O \\ O & (A-B)^{-1} \end{bmatrix} K,$$

in view of Corollary 3.1(i). This leads to the result obtained by Good [26].

For a centrosymmetric matrix of odd order, we have a similar result [24,25]. Such a matrix can be represented as

$$R = \begin{bmatrix} A & \mathbf{x} & BJ \\ \mathbf{y}^\top & r & \mathbf{y}^\top J \\ JB & J\mathbf{x} & JAJ \end{bmatrix},$$

where A and B are matrices of the same size, $\mathbf{x}$ and $\mathbf{y}$ are column vectors, and r is a scalar. Let

$$K = \frac{1}{\sqrt{2}}\begin{bmatrix} I & \mathbf{0} & J \\ \mathbf{0} & 1 & \mathbf{0} \\ -I & \mathbf{0} & J \end{bmatrix};$$

then

$$R = K^{-1}\begin{bmatrix} A+B & 2\mathbf{x} & O \\ \mathbf{y}^\top & r & O \\ O & O & A-B \end{bmatrix} K,$$

and, again by Corollary 3.1(i),

$$R^{-1} = K^{-1}\begin{bmatrix} \begin{bmatrix} A+B & 2\mathbf{x} \\ \mathbf{y}^\top & r \end{bmatrix}^{-1} & O \\ O & (A-B)^{-1} \end{bmatrix} K.$$

The upper left inverse of the 2×2 block matrix can be calculated as before. To be more precise, according to Theorem 2.1(i), if $A+B$ is invertible and

$$t = r - 2\mathbf{y}^\top(A+B)^{-1}\mathbf{x} \neq \mathbf{0},$$

then

$$\begin{bmatrix} A+B & 2\mathbf{x} \\ \mathbf{y}^\top & r \end{bmatrix}^{-1} = \frac{1}{t}\begin{bmatrix} t(A+B)^{-1} + \mathbf{p}\mathbf{q}^\top & -\mathbf{p} \\ -\mathbf{q}^\top & 1 \end{bmatrix},$$

where $\mathbf{p} = 2(A+B)^{-1}\mathbf{x}$ and $\mathbf{q}^\top = \mathbf{y}^\top(A+B)^{-1}$. On the other hand, by Theorem 2.1(ii), if $r \neq 0$ and $M = (A+B) - (2/r)\mathbf{x}\mathbf{y}^\top$ is nonsingular, then

$$\begin{bmatrix} A+B & 2\mathbf{x} \\ \mathbf{y}^\top & r \end{bmatrix}^{-1} = \frac{1}{r^2}\begin{bmatrix} r^2 M^{-1} & -2rM^{-1}\mathbf{x} \\ -r\mathbf{y}^\top M^{-1} & r + 2\mathbf{y}^\top M^{-1}\mathbf{x} \end{bmatrix}.$$

Similarly, a matrix R is skew-centro-Hermitian if $R = -J\bar{R}J$ [22,23]. For such matrices, we have the following corollary.

COROLLARY 4.8. *A matrix is skew-centro-Hermitian of even order if and only if it has the form* $\begin{bmatrix} A & BJ \\ -JB & -JAJ \end{bmatrix}$ *in the all-square partition. Therefore, (2.2)–(2.5) can be used to compute its inverse.*

Since its inverse is skew-centro-Hermitian too, only two blocks of its inverse need to be computed.

A real skew-centro-Hermitian matrix is also called skew-centrosymmetric. An even-order skew-centrosymmetric matrix has the following decomposition [24]:

$$\begin{bmatrix} A & BJ \\ -JB & -JAJ \end{bmatrix} = -K^{-1} \begin{bmatrix} 0 & A-B \\ A+B & 0 \end{bmatrix} K,$$

where K is defined by (4.7). According to Corollary 3.1(ii), its inverse is

$$-K^{-1} \begin{bmatrix} 0 & A-B \\ A+B & 0 \end{bmatrix}^{-1} K = -K^{-1} \begin{bmatrix} 0 & (A+B)^{-1} \\ (A-B)^{-1} & 0 \end{bmatrix} K.$$

For a skew-centrosymmetric matrix R of odd order n, it is always singular and has no inverse. This can be seen by

$$\det R = (-1)^n \det J \cdot \det R \cdot \det J = -\det R.$$

Our formulae are also useful for the other structured matrices. For example, all Hankel matrices are symmetric, and it is natural to use the square diagonal partition and Corollary 4.1 to compute their inverses. For a Hankel matrix of even order, the all-square partition is the best choice, with respect to which it has the form $\begin{bmatrix} A & B \\ B & D \end{bmatrix}$. In this case, all (2.2)–(2.5) can be used. Even in the square off-diagonal partition, off-diagonal blocks of every Hankel matrix are strongly related. In fact, one is a submatrix of the other.

Toeplitz and circulant matrices are persymmetric automatically, so the square off-diagonal partition and Corollary 4.4 are the first choice. For these two types of matrices of even order, the all-square partition is the best to use. In this case, every Toeplitz matrix has the form $\begin{bmatrix} A & B \\ C & A \end{bmatrix}$, and a circulant matrix can be simplified to $\begin{bmatrix} A & B \\ B & A \end{bmatrix}$. The square diagonal partition can be used as well for these matrices, where one of the diagonal blocks is the submatrix of the other.

Ray [27] considers the inverse of a symmetric Toeplitz matrix. Such a matrix is symmetric, persymmetric, and centrosymmetric. Therefore, all our methods, in the three kinds of partitions, in this section can be applied. Ray actually gives three formulae to compute the inverse of such matrix in the all-square partition. All of them can be obtained from our methods, and there are other formulae we can obtain which are not listed in [27].

5. COMPLETION PROBLEMS

In this final section, we give a brief introduction to a more general problem, i.e., the completion problem of a 2×2 block matrix and its inverse. This problem determines if there exists a nonsingular matrix with some known entries so that its inverse has specified elements. To be more precise, supposing

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}, \tag{5.1}$$

and four of A through G are given, our goal is to find all the other block matrices in (5.1). When four of the given blocks are all on the same side of (5.1), it becomes a simple problem to find the inverse matrix, which we have done in Section 2. Thus, it is more interesting when the four given blocks are on different sides of (5.1). It can be three blocks on one side and one on the other, or each side has exactly two blocks known.

Many works can be found on this type of matrix completion problem. Fiedler and Markham [28] study the block completion problem (5.1) where A, B, C, and H are known. For the general transposed partition, they give the necessary and sufficient conditions such that the problem has a solution. Hua [29] completely solves the same completion problem but with a symmetric assumption: given A, B, C, and H, find D, E, F, and G satisfying (5.1) such that both blocked matrices in (5.1) are symmetric. As a special case, he solves the same completion problem with the symmetric positive definite constraint.

Barrett *et al.* [30] consider (5.1) in a general transposed partition with A, D, F, and G known, and give several necessary and sufficient conditions such that it is solvable. This problem is more difficult since it is related to a quadratic matrix equation. Helton *et al.* repeatedly solve this exact problem using the noncommutative software package NCAlgebra [14]. Assisted by the same package, Kronewitter solves a specific 3×3 block matrix completion problem and amazingly obtains 31,000 new theorems. The methodology, programs, and other applications can be found in [15] as well as in their website `http://math.ucsd.edu/~ncalg`.

It is very probable that all the formulae in this paper can be proved by computer techniques independent of traditional human manipulation. One can assert that proper use of a modern computer will dramatically increase the power of theorem proving. In this final remark, we would like to encourage further exploration of this subject.

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