# Bordered Matrices

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ABSTRACT

This paper is a thorough examination of the connections between the kernels of a "bordered matrix"

$$H(z)=\begin{pmatrix} a & v^* & z^* \\ v & H_2 & w \\ z & w^* & b \end{pmatrix}$$

and the kernels of the "largest" principal submatrices

$$H_1=\begin{pmatrix} a & v^* \\ v & H_2 \end{pmatrix}, \qquad H_3=\begin{pmatrix} H_2 & w \\ w^* & b \end{pmatrix},$$

and $H_2$. ($H_1$, $H_2$, and $H_3$ will usually be *not* invertible.) These results on bordered matrices may be used in order to construct invertible "extensions" of certain types of "band" matrices.

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## 1. INTRODUCTION

This paper is a thorough examination of the connections between the kernels of the four "largest" principal submatrices in a "bordered matrix."




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BORDERED MATRIX HYPOTHESES. Let $H_2$ be an $(r-2)\times(r-2)$ hermitian matrix. Let $v$ and $w$ be vectors in $\mathbf{C}^{r-2}$, and let $a$ and $b$ be real numbers. Set

$$H_1=\begin{pmatrix} a & v^* \\ v & H_2 \end{pmatrix} \quad \text{and} \quad H_3=\begin{pmatrix} H_2 & w \\ w^* & b \end{pmatrix},$$

and let $H(z)$ be the bordered matrix

$$H(z)=\begin{pmatrix} a & v^* & z^* \\ v & H_2 & w \\ z & w^* & b \end{pmatrix}.$$

Let $\delta_1$, $\delta_2$, $\delta_3$, and $\delta(z)$ denote the dimensions of the kernels of $H_1$, $H_2$, $H_3$, and $H(z)$. Let $\nu_1$, $\nu_2$, $\nu_3$, and $\nu(z)$ denote the negativities of $H_1$, $H_2$, $H_3$, and $H(z)$.

NOTATION. The natural injection maps $I_i$: Domain of $H_i \to$ Domain of $H(z)=\mathbf{C}^r$, $i=1$, 2 and 3, are $I_1(v)=(v,0)$, $I_2(v)=(0,v,0)$ and $I_3(v)=(0,v)$. We associate earh $\operatorname{Ker} H_i$ with its image space $I_i(\operatorname{Ker} H_i)$. Therefore we will write $\operatorname{Ker} H_1=\operatorname{Ker} H_2$ as a shorthand notation for $I_1(\operatorname{Ker} H_1)=I_2(\operatorname{Ker} H_2)$.

DEFINITION. $H(z)$ is called a *one-step extension* of $H(0)$.

These bordered matrices have been much studied in [3], [4], [5], [6], and [7] when $H_1$ and $H_2$ are invertible matrices. Those results were the basic tool for constructing (in those papers) invertible hermitian extensions of "band" matrices, often with special properties.

In this paper, $H_1$, $H_2$, and $H_3$ will usually be *not* invertible. In Section 3, we will list many of the possibilities for $\delta(z)$. The common threads (and results) of Section 3 are collected in the following two theorems.

THEOREM 1.1 (On bordered matrices). *Given the Bordered Matrix Hypotheses, there is a number $z$ such that*

$$\delta(z)\leqslant \max\{\delta_1,\delta_3\} \quad \textit{and} \quad \delta(z)+\nu(z)=\max\{\delta_1+\nu_1,\delta_3+\nu_3\}. \tag{1.1}$$

*Furthermore, either* $\operatorname{Ker} H_1 \supset \operatorname{Ker} H(z)$ *or* $\operatorname{Ker} H_3 \supset \operatorname{Ker} H(z)$; *also*

$$\delta(z)\leqslant \delta_2. \tag{1.2}$$

*When* $|\delta_1 - \delta_3| = 1$, *then*

$$\delta(z) = \min\{\delta_1, \delta_3\}.$$

*When* $|\delta_1 - \delta_3| = 2$, *then*

$$\delta(z) = \text{average}\{\delta_1, \delta_3\}.$$

*Also*

$$\delta(z) = \max\{\delta_1, \delta_3\} \quad \Rightarrow \quad \operatorname{Ker} H(z) = \bigcap_{i=1}^{3} \operatorname{Ker} H_i.$$

*Furthermore, all these results are valid for all complex numbers $z$ in some half plane $P$ in the complex plane. Also, when $H_1$ and $H_3$ are both real matrices, then this half plane $P$ will include a half line of the real axis.*

REMARK. By collecting the hypotheses and conclusions of Lemmas 3.3 part (b), 3.5 and 3.6 part (c) the following theorem will be established.

THEOREM 1.2. *Given the Bordered Matrix Hypotheses. If*

$$\delta_2 \leqslant \delta_1 \quad \textit{and} \quad \delta_2 \leqslant \delta_3,$$

*then there is a number $z_0$ such that*

$$\pi(z_0) = \max\{\pi_1, \pi_3\} \quad \textit{and} \quad \gamma(z_0) = \max\{\nu_1, \nu_3\} \quad \textit{and}$$

$$\delta_1 \leqslant \delta(z_0) \quad \textit{and} \quad \delta_3 \leqslant \delta(z_0).$$

*Furthermore when $H_1$ and $H_3$ are real matrices, then the number $z_0$ may be chosen to be a real number also.*

DEFINITION. A matrix with all zeros off the main diagonal and the first $m$ pairs of superdiagonals is called a *band matrix* with *bandwidth* $m$. We say that an $n \times n$ matrix $R = (r_{jk})$ is an *m-band* matrix if $r_{rj} = 0$ for all $|k - j| > m$, and an $n \times n$ hermitian matrix $F = (f_{jk})$ is an *extension* of such a matrix $R$ if $f_{jk} = r_{jk}$ for all $|k - j| \leqslant m$.

As simple consequences of Theorem 1.1, we will establish Theorem 1.4 on extensions of band matrices. More results on how to construct invertible extensions of band matrices will be presented in a sequel to this paper ([8]).

DEFINITION. The *inertia* of a hermitian matrix $H$ is a triple $\operatorname{In} H = (\pi, \nu, \delta)$ consisting of the numbers of positive, negative, and zero eigenvalues of $H$. We let $\pi(H)$, $\nu(H)$, and $\delta(H)$ denote the three coordinates of $\operatorname{In} H$.

DEFINITION. A *maximal hermitian submatrix* within an $m$-band $n \times n$ matrix is a (hermitian) $(m+1)\times(m+1)$ submatrix that is within the band. For an $m$-band $n \times n$ matrix $R$, let $R_1, R_2, \ldots, R_{n-m+1}$ denote the $n-m+1$ maximal hermitian submatrices ordered from the upper left corner.

THEOREM 1.3. ([8]) *Let $R$ be an $m$-band $n \times n$ hermitian matrix. Suppose that the $r^{th}$ maximal submatrix $R_r$ is invertible, for a positive integer $r$, $m+5 \leqslant 2r \leqslant n-m$. Then $R$ has an invertible hermitian completion.*

DEFINITION. Suppose that a hermitian band matrix $R$ has been filled in to a hermitian extension $F$ by a sequence of one-step extensions consistent with Theorem 1.1 on bordered matrices. We shall say that $F$ was constructed from $R$ by the *standard procedure*.

THEOREM 1.4. *Suppose that a hermitian matrix $F$ was constructed from a hermitian $m$-band $n \times n$ matrix $R$ by the standard procedure. Then*

$$\nu(F)+\delta(F)=\max\{\nu(R_i)+\delta(R_i),\ i=1,2,\ldots,n-m\}. \tag{1.3}$$

*Also, there is an index $j$ such that* $\operatorname{Ker} R_j \supset \operatorname{Ker} F$.
*If $2r+1=n-m$ then*

$$\delta(F)\leqslant\delta(R_{r+1}). \tag{1.4}$$

*Furthermore, when the band matrix $R$ is a real matrix, then the extension $F$ will also be real matrix.*

This paper is organized as follows. In Section 2, we state Theorem 2.1 (from a companion paper [2]), which describes some aspects of the connections between the inertias of a hermitian matrix and a principal submatrix. Theorem 2.1 is specialized as Corollary 3.2 on "semibordered" matrices, which is then used as the main tool for establishing many specific properties of bordered matrices (Lemmas 3.3–3.9). Theorem 1.4 is established in Section 4, largely as a corollary of Theorem 1.1 on bordered matrices.

## 2. BACKGROUND

We begin this section by stating part of our Theorem 1.2 of [2], namely:

THEOREM 2.1. *Let $H_7$ be a principle $(n-r)\times(n-r)$ submatrix of a hermitian $n\times n$ matrix $H$. We set*

$$\Delta = \operatorname{Dim}\operatorname{Ker} H_7 - \operatorname{Dim}(\operatorname{Ker} H_7 \cap \operatorname{Ker} H),$$

$$\Delta^* = \operatorname{Dim}\operatorname{Ker} H - \operatorname{Dim}(\operatorname{Ker} H_7 \cap \operatorname{Ker} H).$$

*Then*

(a) $\pi(H) \geqslant \pi(H_7) + \Delta$ *and*
(b) $\nu(H) \geqslant \nu(H_7) + \Delta$.

*Also*

(c) $\Delta + \Delta^* \leqslant r$.

*Proof.* Parts (a) and (b) are from Theorem 1.2 of [2]. Part (c) is an immediate consequence of the inequalities (1.5) of [2]. ■

The major tool used in this paper will be Corollaries 3.1 and 3.2, which are consequences of Theorem 2.1 when $r = 1$.

EXAMPLE 2.2. Let us examine the matrix

$$H(z) = \begin{pmatrix} 1 & 1 & z^* \\ 1 & 1 & 2 \\ z & 2 & 4 \end{pmatrix}$$

and its submatrices

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad H_3 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

We observe that $\delta(z) = 0$ when $z \neq 2$, and $\delta(2) = 2$. Also

$$\operatorname{Ker} H_1 = \operatorname{Span}\{(1, -1)^T\} \subset R^2 \times 0 \text{ and } \operatorname{Ker} H_3 = \operatorname{Span}\{(2, -1)^T\} \subset 0 \times R^2.$$

We observe that $\operatorname{Ker} H(2) = \operatorname{Ker} H_1 + \operatorname{Ker} H_3$, but $\operatorname{Ker} H(z) = 0$ for all $z \neq 2$. Therefore either the choice of $z$ will preserve both kernels of $H_1$ and $H_3$ or both of these kernels will disappear. Thus the "fate" of these kernels is linked. This is in contrast to the situation in [1], where we were able to freely pick and choose which parts of the kernels we wished to preserve and to destroy. Afterwards, we were still able to choose any inertia for the hermitian extension of a block diagonal hermitian matrix, which is consistent with Theorem 2.1. The result in [1] was:

THEOREM 2.3. *Given s Hermitian matrices* $S_1, S_2, \dots, S_s$, *with inertias* $(\pi_i, \nu_i, \delta_i)$. *Let each* $n_i = \pi_i + \nu_i + \delta_i$, *and let* $n = \sum n_i$. *Let S be a block diagonal matrix with* $S_1, S_2, \dots, S_s$ *as the blocks on the main diagonal. Choose any subspaces* $U_i$ *of* $\operatorname{Ker} S_i$, *and let* $\Delta_i$ *be the codimension of* $U_i$ *in* $\operatorname{Ker} S_i$. *Choose any nonnegative integers* $\pi$, $\nu$, *and* $\delta$ *such that* $n = \pi + \nu + \delta$ *and*

$$\pi \geqslant \max\{\pi_i + \Delta_i\}, \qquad \nu \geqslant \max\{\nu_i + \Delta_i\}, \quad \text{and} \quad \delta \geqslant \Sigma(\delta_i - \Delta_i).$$

*Then there is a hermitian extension F of S with* $\operatorname{In} F = (\pi, \nu, \delta)$, *and each* $U_i = \operatorname{Ker} F \cap \operatorname{Ker} S_i$.

Additional results on hermitian extension of band matrices appear in [3], [4], [5], [6], and [7].

## 3. ON THE KERNELS OF BORDERED MATRICES

The organization of this section is this: First we specialize Theorem 2.1 to the case $r = 1$ as Corollaries 3.1 and 3.2. We then use these corollaries repeatedly as we examine a collection of cases of bordered matrices. These results are then collected (with the aid of Claim 3.10) in Theorem 1.1 on bordered matrices. In turn, Theorem 1.1 is the basis for the proofs of Theorem 1.4.

COROLLARY 3.1. *Let* $H_1$ *and* $H_2$ *be as in the Bordered Matrix Hypotheses (stated in Section* 1).

(a) *Then either* $\operatorname{Ker} H_1 \supsetneqq \operatorname{Ker} H_2$ *or* $\operatorname{Ker} H_2 \supsetneqq \operatorname{Ker} H_1$ *or* $\operatorname{Ker} H_1 = \operatorname{Ker} H_2$. *Also* $|\delta_1 - \delta_2| \leqslant 1$.

(b) *If* $\delta_1 = \delta_2$, *then* $\operatorname{Ker} H_1 = \operatorname{Ker} H_2$.

(c) *If* $\delta_1 > \delta_2$, *then* $\operatorname{Ker} H_1 \supsetneqq \operatorname{Ker} H_2$ *and* $\operatorname{In} H_1 = \operatorname{In} H_2 + (0,0,1)$.

(d) *If* $\delta_1 < \delta_2$, *then* $\operatorname{Ker} H_2 \supsetneqq \operatorname{Ker} H_1$ *and* $\operatorname{In} H_1 = \operatorname{In} H_2 + (1,1,-1)$, *and* $v$ *is not in the image space* $H_2(C^{r-2})$.

(e) *If* $\delta_1 = \delta_2$ *or* $\delta_1 = \delta_2 + 1$, *then* $v^* \in (\operatorname{Ker} H_2)^{\perp}$.

*Proof.* Theorem 2.1, part (c), tells us that $\Delta + \Delta^* \leqslant 1$. Therefore, in going from $H_2$ to $H_1$ the kernel can gain or lose a vector, but not both. This establishes (a). Except for the "also" part of (d), the next three results listed are immediate consequences of (a). Then

$$\operatorname{In} H_1 = \operatorname{In} H_2 + (1,1,-1) \quad \Rightarrow \quad \operatorname{Rank} H_1 = 2 + \operatorname{Rank} H_2.$$

This forces $v$ to *not* be in the image space $H_2(C^{r-2})$. Finally, (b) and (c) imply (e). ■

REMARK. Note that parts (b), (c), and (d) include all cases; therefore they are all if-and-only-if statements. Hence:

COROLLARY 3.2 (On "semibordered" matrices). *Let $H_1$ and $H_2$ be as in the Bordered Matrix Hypotheses.*

(a) *Then* $|\delta_1 - \delta_2| \leqslant 1$.
(b) $\delta_1 = \delta_2 \Leftrightarrow \operatorname{Ker} H_1 = \operatorname{Ker} H_2$.
(c) $\delta_1 = \delta_2 + 1 \Leftrightarrow \operatorname{Ker} H_1 \supset \operatorname{Ker} H_2 \Leftrightarrow \operatorname{In} H_1 = \operatorname{In} H_2 + (0,0,1)$.
(d) $\delta_1 = \delta_2 - 1 \Leftrightarrow \operatorname{Ker} H_2 \supset \operatorname{Ker} H_1 \Leftrightarrow \operatorname{In} H_1 = \operatorname{In} H_2 + (1,1,-1)$.

Thus, the connection between the dimensions of the kernels determines the connection between the kernels. We will use this to show (in Lemmas 3.3–3.9) that the connections between $\delta_1$, $\delta_2$, and $\delta_3$ almost determine $\delta(z)$.

OBSERVATION. All the results of Corollaries 3.1 and 3.2 about $\{H_1, H_2\}$ also are applicable to each of these ordered pairs of matrices: $\{H(z), H_1\}$, $\{H(z), H_3\}$, and $\{H_3, H_2\}$.

We will implicitly use this observation often in the proofs in this section. The reader might review Example 2.2 now, because it is a specific example of the next lemma.

LEMMA 3.3. *Suppose that $\delta_1 = \delta_2 + 1$ and $\delta_3 = \delta_2$ or $\delta_1$. Then there is a single number $z_0$ such that*

(a) *for all $z \neq z_0$,* $\operatorname{In} H(z) = \operatorname{In} H_1 + (1,1,-1)$; *also* $\delta(z) = \delta_2 = \delta_1 - 1$, $\nu(z) = \nu_1 + 1$, *and* $\operatorname{Ker} H_1 \supset \operatorname{Ker} H(z) = \operatorname{Ker} H_2$;
(b) $\operatorname{In} H(z_0) = \operatorname{In} H_3 + (0,0,1)$; *also* $\operatorname{Ker} H(z_0) \supset \operatorname{Ker} H_3 \cup \operatorname{Ker} H_2$.

*Proof.* The hypotheses imply that Rank $H_1 = \text{Rank } H_2$. Therefore, there is a vector $k$ such that

$$\begin{pmatrix} a \\ v \end{pmatrix} = \begin{pmatrix} v^* \\ H_2 \end{pmatrix} k.$$

Actually, the general solution to the equation $v = H_2 k$ is $k + \operatorname{Ker} H_2$. Corollary 3.1, part (e), tells us that $w^*u = 0 = v^*u$ for all $u \in \operatorname{Ker} H_2$. we may set

$$z_0 = w^*k = w^*(k + \operatorname{Ker} H_2).$$

We observe that

$$H_1\begin{pmatrix} -1 \\ k \end{pmatrix} = 0, \quad \text{but} \quad H(z)\begin{pmatrix} -1 \\ k \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -z + w^*k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z_0 - z \end{pmatrix}.$$

Thus

$$H(z)(-1, k, 0)^T = (0, 0, z_0 - z)^T. \tag{3.1}$$

Thus $\operatorname{Ker} H_1$ is not contained in $\operatorname{Ker} H(z)$, and $\operatorname{Ker} H_1 \neq \operatorname{Ker} H(z)$ when $z \neq z_0$. This rules out parts (b) and (c) of Corollary 3.1. Therefore part (d) must apply. This establishes part (a) of this lemma.

When $\delta_1 = \delta_3$, then the relationship between $H_3$ and $H_2$ is the same as for $H_1$ and $H_2$ above. Therefore, there is a vector $k_2$ such that

$$\begin{pmatrix} w \\ b \end{pmatrix} = \begin{pmatrix} H_2 \\ w^* \end{pmatrix} k_2.$$

The general solution to the equation $w = H_2 k_2$ is $k_2 + \operatorname{Ker} H_2$. Here $H(z)(0, k_2, -1)^T = (v^*k_2 - z^*, 0, 0)^T$. We calculate

$$v^*k_2 = k^*H_2k_2 = k^*w.$$

Hence $z_0 = w^*k$ implies that $z_0^* = v^*k_2$. Thus $H(z_0)(-1, k, 0)^T = 0$ and $H(z_0)(0, k_2, -1)^T = 0$. Thus the kernel of $H(z_0)$ has *two* linearly independent vectors which are not in $\operatorname{Ker} H_2$. This and part (c) of Theorem 2.1 (with $r = 2$) implies that $\delta(z_0) = 2 + \delta_2$. This and Corollary 3.2 will establish part (b) of this lemma when $\delta_1 = \delta_3$.

When $\delta_1 - 1 = \delta_3 = \delta_2$, then $\operatorname{Ker} H_2 = \operatorname{Ker} H_3$. Equation (3.1) shows us that $(-1, k, 0)^T \in \operatorname{Ker} H(z_0)$. But this vector $(-1, k, 0)^T$ cannot also be in $\operatorname{Ker} H_3$. Hence Corollary 3.2 will establish part (b) of this lemma when $\delta_1 - 1 = \delta_3$. ∎

LEMMA 3.4. *Suppose that $\delta_3 = \delta_1 + 2$. Then*

$$\operatorname{In} H(z) = \operatorname{In} H_1 + (0,0,1) = \operatorname{In} H_2 + (1,1,0) = \operatorname{In} H_3 + (1,1,-1).$$

*Also* $\delta(z) = \delta_3 - 1 = \delta_2 = \delta_1 + 1$.

*Proof.* Corollary 3.1, part (a), tells us that both $\delta(z)$ and $\delta_2$ must be within 1 or both $\delta_1$ and $\delta_3$. Therefore, the only choice is $\delta(z) = \delta_2 = \delta_1 + 1$. This plus Corollary 3.1, parts (c) and (d), will establish this lemma. ∎

LEMMA 3.5. *Suppose $\delta_1 = \delta_2 = \delta_3$ and $\nu_1 = \nu_2 + 1 = \nu_3 + 1$. Then*

$$\operatorname{In} H(z) = \operatorname{In} H_1 + (1,0,0) = \operatorname{In} H_2 = (1,1,0) = \operatorname{In} H_3 + (0,1,0),$$

*and*

$$\operatorname{Ker} H(z) = \operatorname{Ker} H_1 = \operatorname{Ker} H_2 = \operatorname{Ker} H_3 \qquad \textit{for all } z.$$

*Proof.* These hypotheses imply that $H_1$ and $H_3$ have only an additional negative eigenvalue and positive eigenvalue, respectively. This together with Cauchy's interlacing theorem or Theorem 2.1, forces $H(z)$ to have both an additional negative eigenvalue and an additional positive eigenvalue. This establishes the inertia equations of this lemma. Corollary 3.1, part (a), applied three times, will establish the equations for the kernels. ∎

LEMMA 3.6. *Suppose $\delta_1 = \delta_2 = \delta_3$ and $\nu_1 = \nu_3$. Then there is a circle $C$ in the complex plane such that*

(a) $\delta(z) = \delta_1$ *and* $\nu(z) = \nu_1$ *and* $\operatorname{Ker} H(z) = \operatorname{Ker} H_1 = \operatorname{Ker} H_2 = \operatorname{Ker} H_3$ *for all complex numbers $z$ outside $C$;*

(b) $\delta(z) = \delta_1$, $\pi(z) = \pi_1$, *and* $\operatorname{Ker} H(z) = \operatorname{Ker} H_1 = \operatorname{Ker} H_2 = \operatorname{Ker} H_3$ *for all complex numbers $z$ inside $C$;*

(c) $\delta(z) = 1 + \delta_1$ *for all complex numbers $z$ on $C$.*

*Proof.* There is a unitary matrix $U$ and an invertible matrix $H_2'$ such that

$$U^*H_2U = \begin{pmatrix} H_2' & 0 \\ 0 & 0 \end{pmatrix}.$$

Corollary 3.1, part (b), implies that $\operatorname{Ker} H_1 = \operatorname{Ker} H_2 = \operatorname{Ker} H_3$.

We shall use the change-of-coordinate matrix $U_1 = (1 \oplus U \oplus 1)$ on $H(z)$ in order to simultaneously separate out the "zero" parts of the three $H_i$'s.

Let $(v',0) = U^*v$ and $(w',0) = U^*w$. Then

$$U_1^*H(z)U_1 = \begin{pmatrix} a & v'^* & 0 & z^* \\ v' & H_2' & 0 & w' \\ 0 & 0 & 0 & 0 \\ z & w'^* & 0 & b \end{pmatrix}.$$

We may drop the zero rows and columns in order to obtain a bordered matrix of *invertible* matrices; we set

$$H_1' = \begin{pmatrix} a & v'^* \\ v' & H_2' \end{pmatrix} \quad \text{and} \quad H_3' = \begin{pmatrix} H_2' & w' \\ w'^* & b \end{pmatrix}$$

and let $H'(z)$ be the bordered matrix

$$H'(z) = \begin{pmatrix} a & v'^* & z^* \\ v' & H_2' & w' \\ z & w'^* & b \end{pmatrix}.$$

The circle $C$ for this bordered matrix $H'(z)$ is provided by Ellis, Gohberg, and Lay's Theorem 1.1, part (b), of [4]. Since the unprimed $H$'s are just direct sums of the $H'$'s with zero matrices, the circle $C$ also works for this lemma. ∎

LEMMA 3.7. *Suppose that $\{v, w\}$ is a linearly independent set whose span meets the image space $H_2(\mathbf{C}^{r-2})$ only at $\mathbf{0}$. Then*

$$\operatorname{In} H(z) = \operatorname{In} H_2 + (2,2,-2) \qquad \textit{for all } z. \tag{3.2}$$

*Also*

$$\operatorname{In} H_1 = \operatorname{In} H_3 = \operatorname{In} H_2 + (1,1,-1)$$

*and* $\operatorname{Ker} H(z) = \operatorname{Ker} H_1 \cap \operatorname{Ker} H_3$.

*Proof.* Let $M$ be the matrix such that $M^* = (v, H_2, w)$. The hypotheses imply that

$$\operatorname{Rank} M = \operatorname{Rank} M^* = 2 + \operatorname{Rank} H_2.$$

Hence

$$2 = \operatorname{Dim}\operatorname{Ker} H_2 - \operatorname{Dim}\operatorname{Ker} M.$$

In the context of Theorem 2.1, considering $H_2$ as a submatrix of $H(z)$, we note that $\Delta = 2$. Then Theorem 2.1 will establish Equation (3.2). The other conclusion of this lemma is established in the same manner. ■

LEMMA 3.8. *Suppose that $\delta_2 = \delta_1 = \delta_3 + 1$. Then, for all $z$,*

$$\operatorname{Ker} H_1 \supset \operatorname{Ker} H(z) \quad \textit{and} \quad \operatorname{In} H(z) = \operatorname{In} H_1 + (1, 1, -1). \tag{3.3}$$

*Also $\nu(z) + \delta(z) = \nu_1 + \delta_1 \geqslant \nu_2 + \delta_2 = \nu_3 + \delta_3$ and $\delta(z) = \delta_3$.*

*Proof.* Corollary 3.1, part (d), provides

$$\operatorname{In} H_3 = \operatorname{In} H_2 + (1, 1, -1). \tag{3.4}$$

This and Corollary 3.1, part (a), provide $\operatorname{Ker} H_1 = \operatorname{Ker} H_2 \supset \operatorname{Ker} H_3$.

Thus going from $H_2$ to $H_3$ results in the loss of some vector $u$ from $\operatorname{Ker} H_2$. But then this same vector must be lost from $\operatorname{Ker} H_1$ in going from $H_1$ to $H(z)$. Therefore Corollary 3.2, parts (b) and (c), do not apply to $H(z)$ and $H_1$. Only part (d) remains, which implies Equation (3.3). These Equations (3.3) and (3.4), together with $\nu_1 \geqslant \nu_2$, will establish the remainder of this lemma. ■

LEMMA 3.9. *Suppose that both $v$ and $w$ miss the image space $H_2(\mathbf{C}^{r-2})$, and that the* Span$\{v, w\}$ *meets $H_2(\mathbf{C}^{r-2})$ in a line. Then there are two open half planes $P_+$ and $P_-$ whose closures meet in a line $l$, such that:*

(a) $\operatorname{In} H(z) = \operatorname{In} H_1 + (1,0,0)$ *and* $\operatorname{Ker} H(z) = \operatorname{Ker} H_1$ *for all complex numbers $z$ in $P_+$,*

(b) $\operatorname{In} H(z) = \operatorname{In} H_1 + (0,1,0)$ *and* $\operatorname{Ker} H(z) = \operatorname{Ker} H_1$ *for all complex numbers $z$ in $P_-$, and*

(c) $\operatorname{In} H(z) = \operatorname{In} H_1 + (0,0,1)$ *and* $\operatorname{Ker} H(z) \supset \operatorname{Ker} H_1$ *for all complex numbers $z$ on the line $l$.*

*Also,* $\operatorname{In} H_1 = \operatorname{In} H_2 + (1, 1, -1)$ *and* $\operatorname{Ker} H_1 \subset \operatorname{Ker} H_2$ *(for all $z$).*

*Furthermore, when $H(0)$ is a real matrix, then this line $l$ will meet the real axis at a unique real number.*

Before proving this lemma, we will examine a specific example.

EXAMPLE. Let us consider the real matrix:

$$H(z)=\begin{pmatrix}0&1&0&z\\1&1&2&0\\0&2&4&1\\z&0&1&0\end{pmatrix}.$$

We calculate that $\operatorname{Det} H(z) = -4z+1$ and $\operatorname{Det} H_1 = -4$. Since the determinant equals the product of the eigenvalues, we see that $\operatorname{In} H_1 = (2,1,0)$. Going from $H_1$ to $H(z)$ will add a positive, negative, or zero eigenvalue precisely when the determinants of $H(z)$ and $H_1$ have the same sign, or $\operatorname{Det} H(z) = 0$, or the determinants of $H(z)$ and $H_1$ have opposite signs, respectively. These cases occur precisely when $4z>1$, $4z=1$, or $4z<1$, respectively.

*Proof.* There is a unitary matrix $U$ and an invertible matrix $H_4$ such that

$$U^*H_2U=\begin{pmatrix}H_4&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

and such that $U^*v=(v',x,0)^T$ and $U^*w=(w',y,0)^T$, where $x$ and $y$ are complex numbers, $x\neq 0\neq y$.

We shall use the change-of-coordinate matrix $U_1=(1\oplus U\oplus 1)$ on $H(z)$ in order to separate out the zero parts of the $H_i$'s. We obtain this block matrix:

$$U_1^*H(z)U_1=\begin{pmatrix}a&v'^*&x^*&0&z^*\\v'&H_4&0&0&w'\\x&0&0&0&y\\0&0&0&0&0\\z&w'^*&y^*&0&b\end{pmatrix}.$$

We may drop the zero rows and columns in order to obtain these block matrices:

$$H'(z)=\begin{pmatrix}a&v'^*&x^*&z^*\\v'&H_4&0&w'\\x&0&0&y\\z&w'^*&y^*&b\end{pmatrix}\quad\text{and}\quad H_1'=\begin{pmatrix}a&v'^*&x^*\\v'&H_4&0\\x&0&0\end{pmatrix}.$$

We observe that

$$\operatorname{In} H(z) = \operatorname{In} H'(z) + (0,0,\delta_2 - 1) \tag{3.5}$$

and

$$\operatorname{In} H_1 = \operatorname{In} H_1' + (0,0,\delta_2 - 1)$$

Let

$$H_2' = \begin{pmatrix} H_4 & 0 \\ 0 & 0 \end{pmatrix}$$

be the block that is bordered in $H'(z)$. Clearly $\operatorname{Dim}\operatorname{Ker} H_2' = 1$, and this kernel will *not* be "preserved" in $H_1'$ [or in $H'(z)$], because of the $x$ element. Corollary 3.2(d) tells us that

$$\delta(H_1') = \delta(H_2') - 1 = 0 \quad \text{and} \quad \operatorname{In} H_2' + (1,1,-1) = \operatorname{In} H_1'.$$

Hence $\operatorname{Det} H_1' \neq 0$. This, together with Cauchy's interlacing theorem or Theorem 2.1, implies that $\operatorname{In} H'(z) \geqslant \operatorname{In} H_1'$ and that the signs of all but one of the eigenvalues of $H'(z)$ are the same as for $H_1'$. The remaining sign can be determined from the sign of its determinant. Therefore, we note that

$$\operatorname{In} H'(z) = \operatorname{In} H_1' + (1,0,0) = \operatorname{In} H_2' + (2,1,-1) \tag{3.6}$$

when $\operatorname{Det} H'(z)$ and $\operatorname{Det} H_1'$ have the same sign.

We calculate that, as a function of $z$,

$$\operatorname{Det} H'(z) = \operatorname{Re}(a_3 z) + b_3,$$

where $a_3 = 2yx^* \operatorname{Det} H_4 \neq 0$ and $b_3$ is a real number. There is no $zz^*$ term. Therefore, the equation of the desired line $l$ is $0 = \operatorname{Re}(a_3 z) + b_3$. The two half planes $P_+$ and $P_-$ are chosen so that $\operatorname{Det} H'(z)$ and $\operatorname{Det} H_1'$ have the same sign for all $z \in P_+$ and opposite signs for all $z \in P_-$. We calculate further that when $x$ and $y$ are real numbers, this line $l$ will meet the real axis at a unique real number. This and Equations (3.5) and (3.6) will establish the lemma. ∎

CLAIM 3.10. *All the cases when* $\delta_2 \geqslant \delta_1$ *and* $\delta_2 \geqslant \delta_3$ *have been covered.*

*Proof.* By Corollary 3.1, parts (b) and (d), $\operatorname{Ker} H_2 \supseteqq \operatorname{Ker} H_1 \cup \operatorname{Ker} H_3$. As in the proof of Lemma 3.9, a change of coordinates will separate out the kernel of $H_2$. The result is this block matrix:

$$H'(z) = \begin{pmatrix} a & v_1^* & v_2^* & z^* \\ v_1 & H_4 & 0 & w_1 \\ v_2 & 0 & 0 & w_2 \\ z & w_1^* & w_2^* & b \end{pmatrix},$$

where $v_2$ and $w_2$ are vectors in $\mathbf{C}^1$ or $\mathbf{C}^2$, and $H_4$ is an invertible matrix. We note that here there are the following four cases:

(i) $\operatorname{Dim}\operatorname{Span}\{v_2, w_2\} = 2$. This case is the same as Lemma 3.7.
(ii) $v_2 = 0 = w_2$.

SUBCLAIM 3.11. *This case is covered by Lemmas* 3.5 *and* 3.6.

*Proof.* The hypotheses $v_2 = 0 = w_2$ and $H_4$ invertible imply that $\delta_2 \leqslant \delta_1$ and $\delta_2 \leqslant \delta_3$. This and the hypotheses $\delta_2 \geqslant \delta_1$ and $\delta_2 \geqslant \delta_3$ of Claim 3.10 provide $\delta_1 = \delta_2 = \delta_3$.

We will now show that all the possibilities when $\delta_1 = \delta_2 = \delta_3$ have been covered. Cauchy's interlacing theorem or Theorem 2.1 implies that only these three possibi[illegible] exist: (a) $\nu_1 = \nu_2$ or (b)$\nu_1 = \nu_2 + 1 = \nu_3 + 1$ or (c) $\nu_3 = \nu_2 + 1 = \nu_1 + 1$. Switching $\nu_1$ and $\nu_3$ turns (b) into (c). Hence Lemmas 3.5 and 3.6 cover all the possibilities when $\delta_1 = \delta_2 = \delta_3$. ∎

(iii) $w_2 = cv_2 \neq 0$. This case is the same as Lemma 3.9.
(iv) $v_2 = 0 \neq w_2$.

SUBCLAIM 3.12. *This case is covered by Lemma* 3.8.

*Proof.* Corollary 3.2, together with these hypotheses $v_2 = 0 \neq w_2$ and $H_4$ invertible, imply that $\delta_1 \geqslant \delta_2$ and $\delta_2 = \delta_3 + 1$. This and the hypotheses $\delta_2 \geqslant \delta_1$ of Claim 3.10 provide $\delta_1 = \delta_2 = \delta_3 + 1$. Thus the hypotheses of Lemma 3.8 are satisfied. ∎

These four cases establish Claim 3.10. ∎

*Proof of Theorem 1.1 on bordered matrices.* Claim 3.10 and Lemmas 3.3 and 3.4 establish Theorem 1.4. ∎

## 4. SIMPLE DIAGONAL EXTENSIONS

In this section, we will use Theorem 1.1 on bordered matrices in order to establish Theorem 1.4.

DEFINITION. A *simple diagonal extension* of a given $m$-band matrix $R$ is a hermitian extension of $R$ which is an $(m+1)$-band matrix.

When $2r+1=n-m$, the *central matrix* of a $m$-band $n\times n$ matrix is the $(r+1)^{\text{st}}$ maximal submatrix $R_{r+1}$.

*Proof of Equation (1.4).* Let $R$ be an $m$-band matrix with central submatrix $C_2$. Let $R''$ be the simple diagonal completion of the simple diagonal completion of $R$. Let $C$ be the central submatrix of $R''$. The crucial observation is that $C$ is to $C_2$ as $H(z)$ is to $H_2$ since $C$ is the matrix $C_2$ together with a "border". Therefore Equation (1.2) tells us that $\delta(C)\leqslant\delta(C_2)$. Thus, the dimensions of the kernels of the central submatrices cannot increase if we construct all the successive simple diagonal completions according to Theorem 1.1. In this manner, Equation (1.4) is established.

NOTATION. For a band matrix $R$, let

$$\delta^*(R)=\max\{\delta(R_i),\ i=1,2,\dots,n-m\},$$

and let $\nu^*(R)$ be such that

$$\nu^*(R)+\delta^*(R)=\max\{\nu(R_i)+\delta(R_i),\ i=1,2,\dots,n-m\}.$$

LEMMA 4.1. *Given a band matrix $R$, there is a simple diagonal extension $R'$ of $R$ such that*

$$\delta^*(R')\leqslant\delta^*(R)\quad\textit{and}\quad v^*(R')+\delta^*(R')=v^*(R)+\delta^*(R).$$

*Proof.* The new band $R'$ is constructed as the result of a sequence of one-step extensions going down the additional diagonal. We simply use Theorem 1.1 on bordered matrices, including Equation (1.1), at each step. ∎

*Proof of Theorem 1.4.* Equation (1.4) has already been established. Equation (1.3) may be established by simply using the last lemma repeatedly, as one adds pairs of additional diagonals to "fill in" $R$. The conclusion $\operatorname{Ker} R_j \supset \operatorname{Ker} F$, for some index $j$, follows from repeated use of the conclusion $\operatorname{Ker} H_1 \supset \operatorname{Ker} H(z)$ or $\operatorname{Ker} H_3 \supset \operatorname{Ker} H(z)$ of Theorem 1.1 on bordered matrices. ∎

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