Introduction to Bordered Matrices

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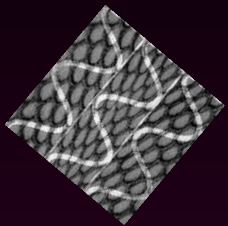
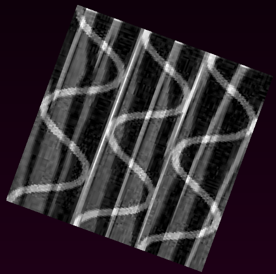
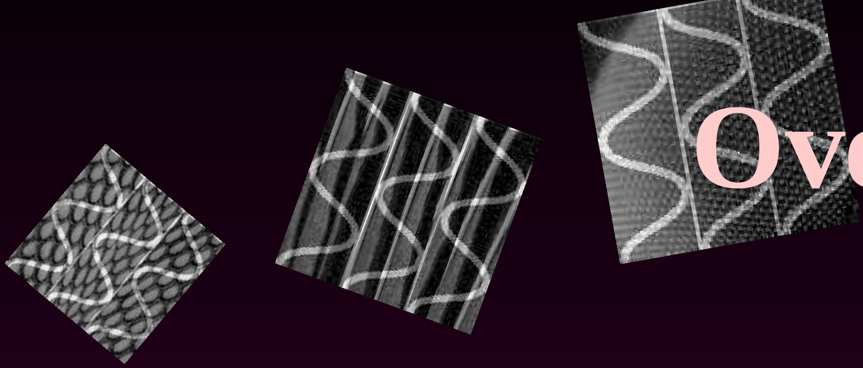




Overview



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  Definition and Interest

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- Bordering a Given Matrix

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- Singular values of Bordered Matrices

  The Singular Value Decomposition

  Singular Value Inequalities

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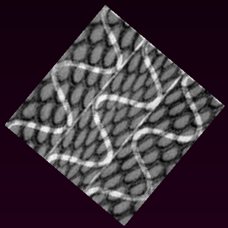
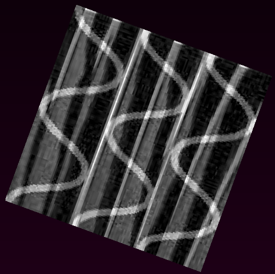
- Defining Systems
- Numerical Methods for Bordered Systems

  BEC, BED, BEM, and BEMW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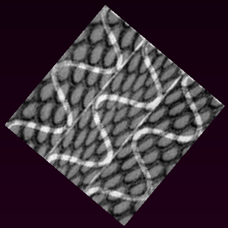

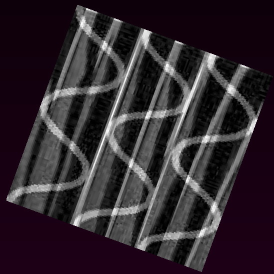

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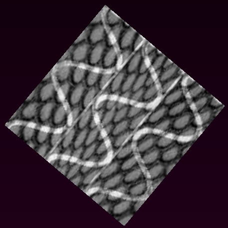
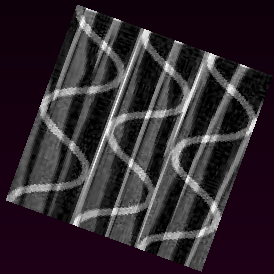
$$M = \begin{pmatrix} A & B \\ C^* & D \end{pmatrix}$$





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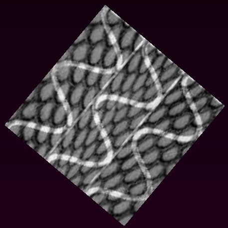
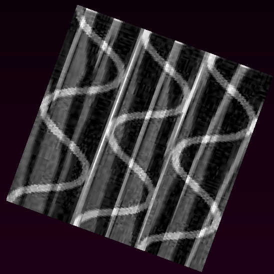
where

- A is $p \times q$
- B is $p \times (n-q)$
- C is $q \times (n-p)$
- D is $(n-p) \times (n-q)$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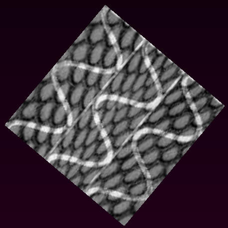
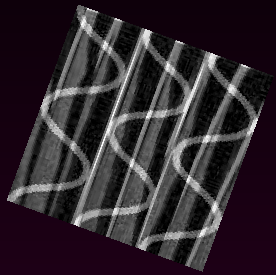
such that M is an $n \times n$ extension of the $p \times q$ matrix A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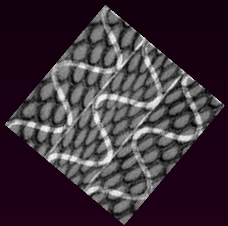
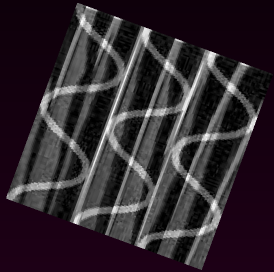
Cleverly chosen extensions M can reveal rank-deficiency of A in a computationally attractive way: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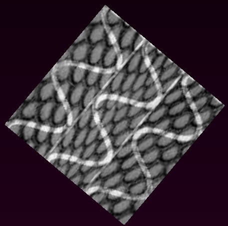
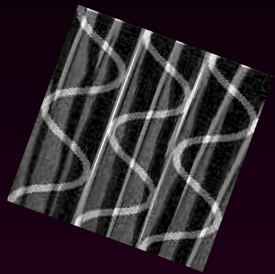
That is, with M itself being nonsingular

Nonsingular Bordering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$$n = p + q - r$$

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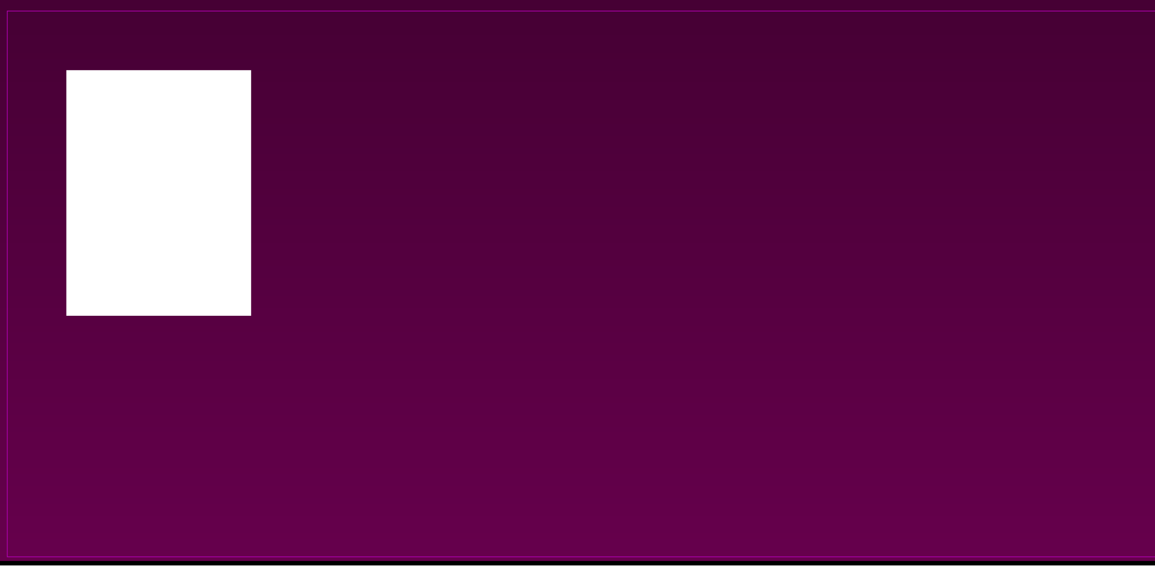

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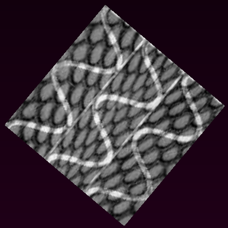
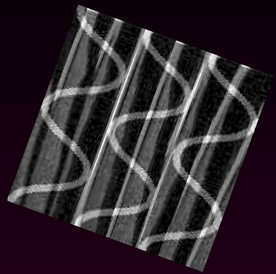


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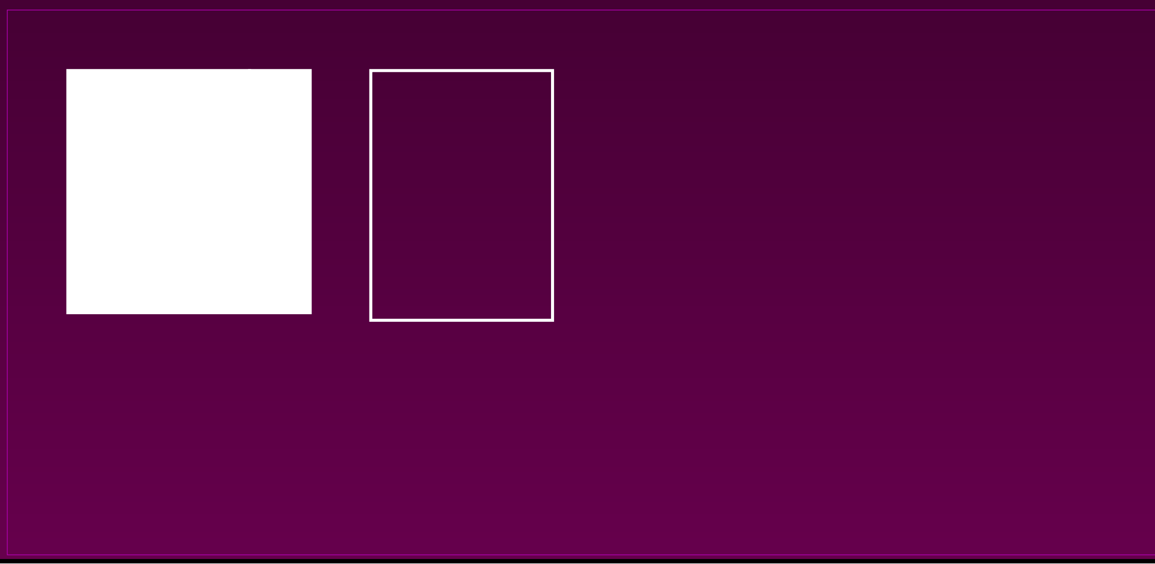

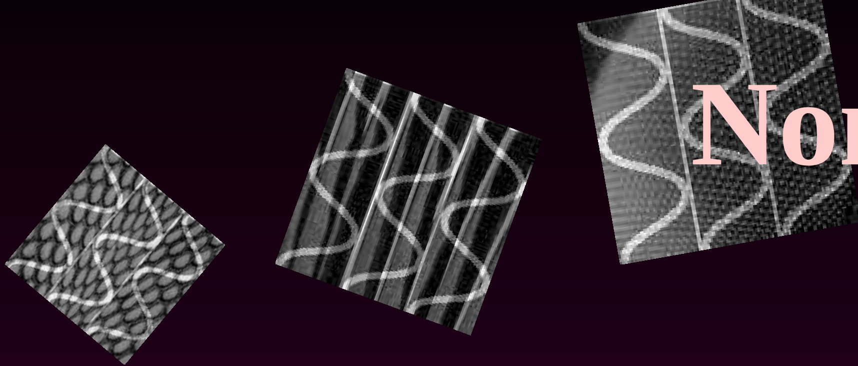

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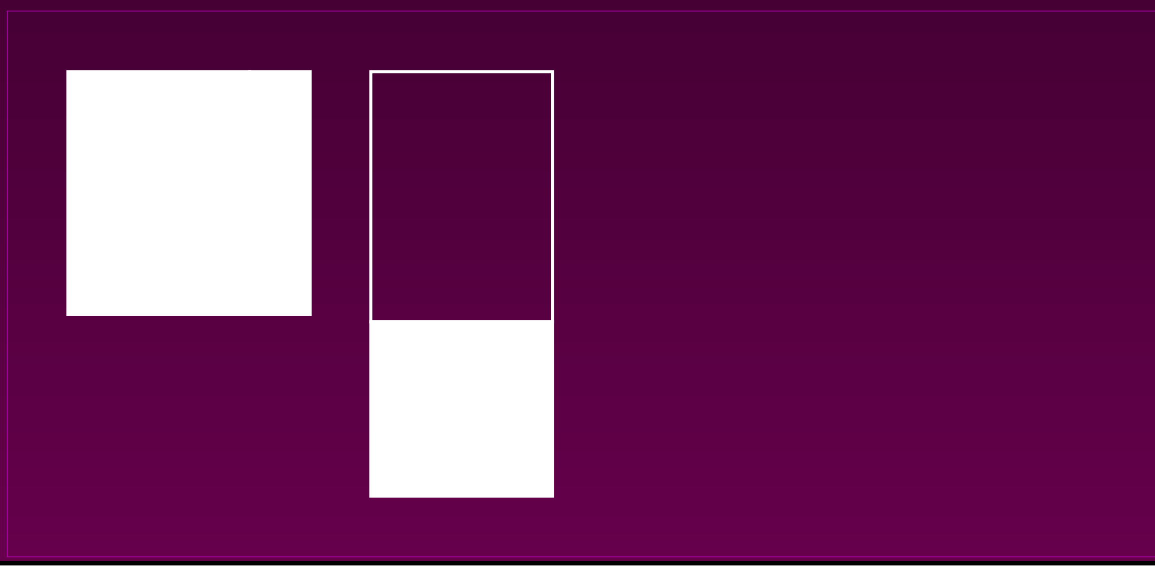

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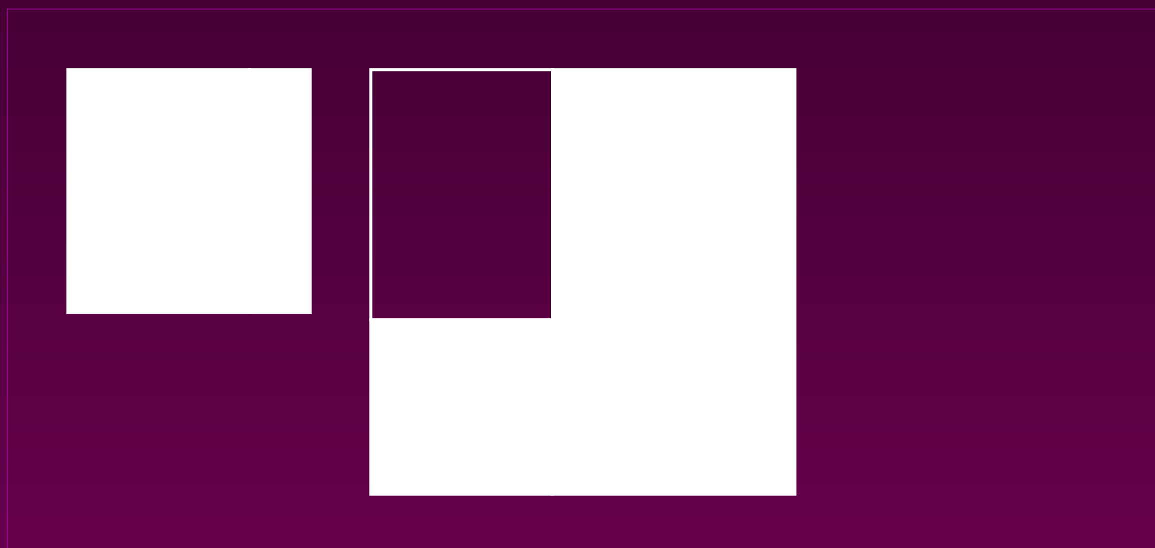

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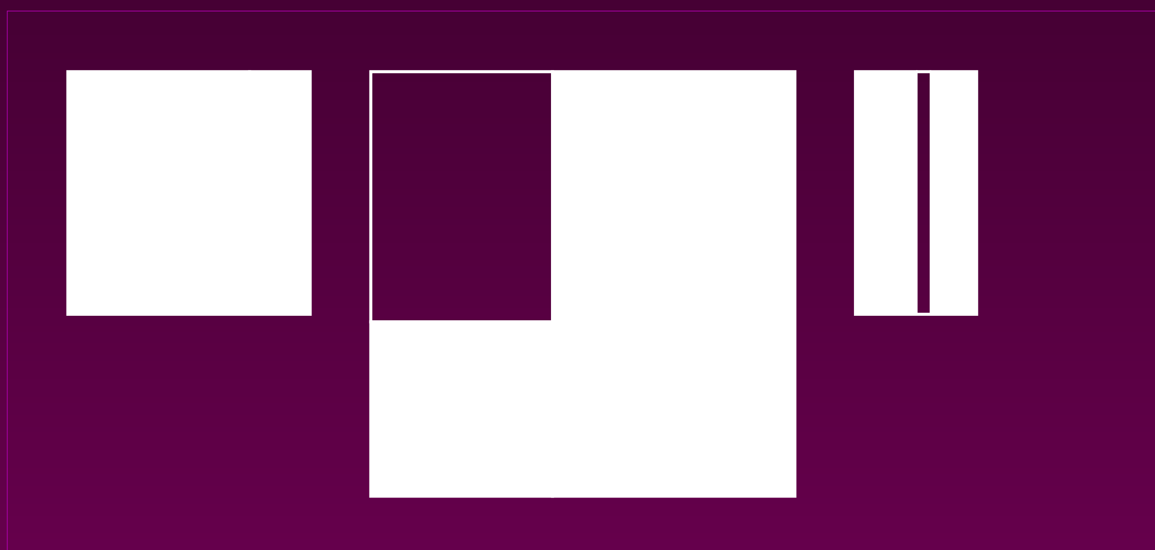

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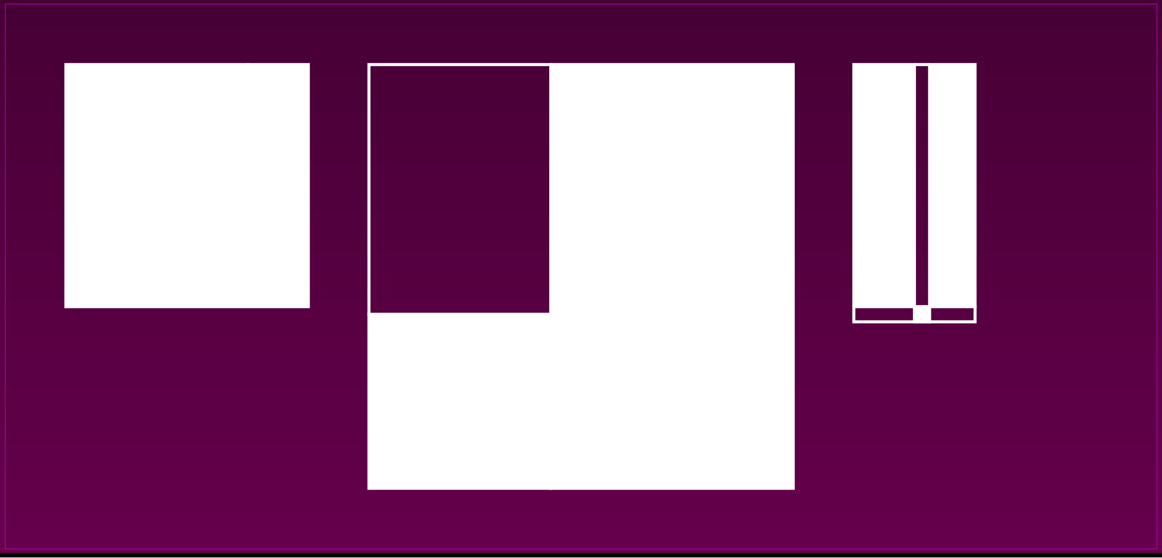

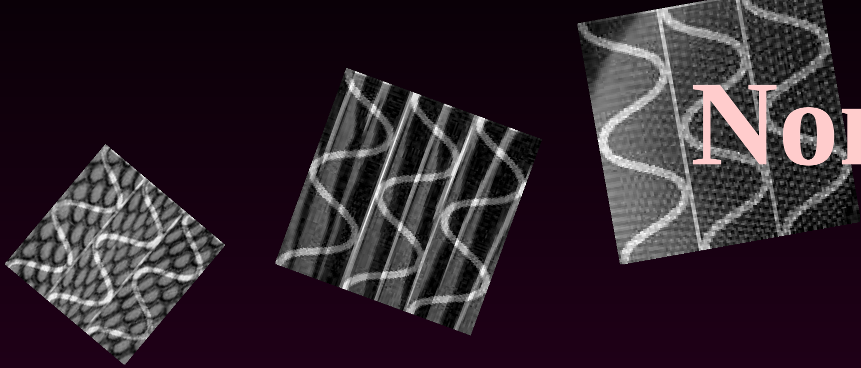

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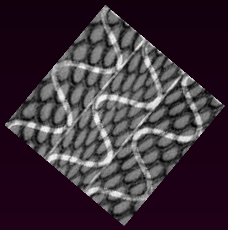

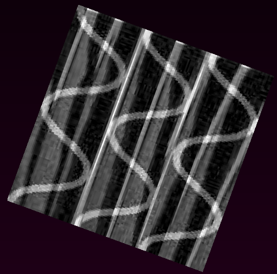

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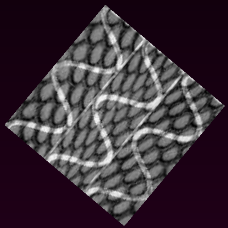
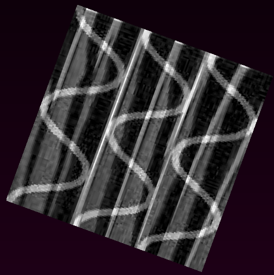
Moreover, for all extensions M of A we have

$$\|M\| \geq \|A\|, \quad \|M^{-1}\| \geq \|A^{\dagger}\|, \quad \kappa(M) \geq \kappa^{\dagger}(A)$$

Equality holds if $D = 0$ and B, C contain orthonormal bases for the kernel and range of A

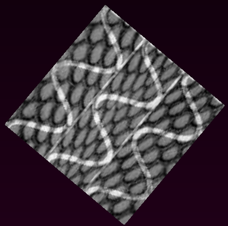
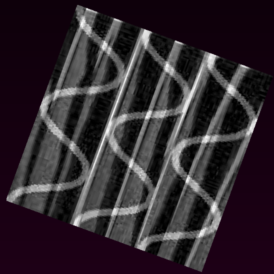
Singular Values


Singular Values

Recall: Each hermitian matrix $A = A^*$ is diagonal on a basis $V = (v_1, \dots, v_n)$ with $V^*V = I$

$$AV = V\Lambda$$

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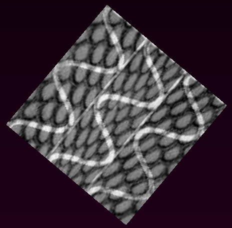
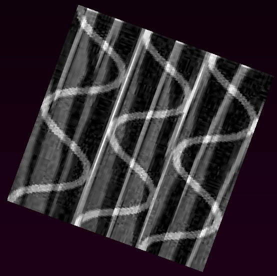
Generalizations of this result for non-hermitian A are the

- Schur decomposition $AQ = QR$ (R upper triangular)
- Singular Value decomposition $AV = U\Sigma$ ($U^*U = I$)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Both imply the above eigendecomposition, but are very different in character