

## GENERALIZED INVERSES OF BORDERED MATRICES\*

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**Abstract.** Several authors have considered nonsingular borderings  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  of  $B$  and investigated the properties of submatrices of  $A^{-1}$ . Under specific conditions on the bordering, one can recover any  $g$ -inverse of  $B$  as a submatrix of  $A^{-1}$ . Borderings  $A$  of  $B$  are considered, where  $A$  might be singular, or even rectangular. If  $A$  is  $m \times n$  and if  $B$  is an  $r \times s$  submatrix of  $A$ , the consequences of the equality  $m + n - \text{rank}(A) = r + s - \text{rank}(B)$  with reference to the  $g$ -inverses of  $A$  are studied. It is shown that under this condition many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. We also consider  $g$ -inverses of the bordered matrix when certain rank additivity conditions are satisfied. It is shown that any  $g$ -inverse of  $B$  can be realized as a submatrix of a suitable  $g$ -inverse of  $A$ , under certain conditions.

**Key words.** Generalized inverse, Moore-Penrose inverse, Bordered matrix, Rank additivity.

**AMS subject classifications.** 15A09, 15A03.

**1. Introduction.** Let  $A$  be an  $m \times n$  matrix over the complex field and let  $A^*$  denote the conjugate transpose of  $A$ . We recall that a generalized inverse  $G$  of  $A$  is an  $n \times m$  matrix which satisfies the first of the four Penrose equations:

$$(1) AXA = A \quad (2) XAX = X \quad (3) (AX)^* = AX \quad (4) (XA)^* = XA.$$

For a subset  $\{i, j, \dots\}$  of the set  $\{1, 2, 3, 4\}$ , the set of  $n \times m$  matrices satisfying the equations indexed by  $\{i, j, \dots\}$  is denoted by  $A\{i, j, \dots\}$ . A matrix in  $A\{i, j, \dots\}$  is called an  $\{i, j, \dots\}$ -inverse of  $A$  and is denoted by  $A^{(i, j, \dots)}$ . In particular, the matrix  $G$  is called a  $\{1\}$ -inverse or a  $g$ -inverse of  $A$  if it satisfies (1). As usual, a  $g$ -inverse of  $A$  is denoted by  $A^-$ . If  $G$  satisfies (1) and (2) then it is called a reflexive inverse or a  $\{1, 2\}$ -inverse of  $A$ . Similarly,  $G$  is called a  $\{1, 2, 3\}$ -inverse of  $A$  if it satisfies (1), (2) and (3). The Moore-Penrose inverse of  $A$  is the matrix  $G$  satisfying (1)-(4). Any matrix  $A$  admits a unique Moore-Penrose inverse, denoted  $A^\dagger$ . If  $A$  is  $n \times n$  then  $G$  is called the group inverse of  $A$  if it satisfies (1), (2) and  $AG = GA$ . The matrix  $A$  has group inverse, which is unique and denoted by  $A^\sharp$ , if and only if  $\text{rank}(A) = \text{rank}(A^2)$ . We refer to [4], [6] for basic results on  $g$ -inverses.

Suppose

$$(1.1) \quad A = \begin{matrix} & \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \end{matrix} & \begin{pmatrix} B & C \\ D & X \end{pmatrix} \end{matrix}$$

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is a partitioned matrix. We say that  $A$  is obtained by bordering  $B$ . We will generally partition a  $g$ -inverse  $A^-$  of  $A$  as

$$(1.2) \quad A^- = \begin{matrix} & p_1 & p_2 \\ q_1 & \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \\ q_2 & \end{matrix}$$

which is in conformity with  $A^*$ .

We say that the  $g$ -inverses of  $A$  have the “block independence property” if for any  $g$ -inverses

$$A_i^- = \begin{pmatrix} E_i & F_i \\ G_i & Y_i \end{pmatrix}, \quad i = 1, 2$$

of  $A$ ,  $\begin{pmatrix} E_1 & F_1 \\ G_1 & Y_2 \end{pmatrix}$ ,  $\begin{pmatrix} E_1 & F_1 \\ G_2 & Y_1 \end{pmatrix}$  etc. are also  $g$ -inverses of  $A$ .

If  $A$  is an  $m \times n$  matrix, then the following function will play an important role in this paper:

$$\psi(A) = m + n - \text{rank}(A).$$

An elementary result is given next. For completeness, we include a proof.

LEMMA 1.1. *If  $B$  is a submatrix of  $A$ , then  $\psi(B) \leq \psi(A)$ .*

*Proof.* Let

$$A = \begin{matrix} & q_1 & q_2 \\ p_1 & \begin{pmatrix} B & C \\ D & X \end{pmatrix} \\ p_2 & \end{matrix}$$

Then

$$\begin{aligned} \text{rank}(A) &\leq \text{rank} \begin{pmatrix} B & C \\ D & X \end{pmatrix} + \text{rank} \begin{pmatrix} D & X \end{pmatrix} \\ &\leq \text{rank}(B) + \text{rank}(C) + p_2 \\ &\leq \text{rank}(B) + q_2 + p_2. \end{aligned}$$

From this inequality, we get  $\psi(B) \leq \psi(A)$ .  $\square$

Note that a matrix  $B$  with  $\text{rank}(B) = r$  can be completed to a nonsingular matrix  $A$  of order  $n$  if and only if  $\psi(B) \leq n$  [10, Theorem 5]. As another example of a result concerning  $\psi$ , if

$$A = \begin{matrix} & q_1 & q_2 \\ p_1 & \begin{pmatrix} B & C \\ D & O \end{pmatrix} \\ p_2 & \end{matrix}$$

is a nonsingular matrix of order  $n$ ,  $n = p_1 + p_2 = q_1 + q_2$ , then  $A^{-1}$  is of the form

$$A^{-1} = \begin{matrix} & p_1 & p_2 \\ q_1 & \begin{pmatrix} E & F \\ G & O \end{pmatrix} \\ q_2 & \end{matrix}$$

if and only if  $\psi(B) = \psi(A)$ . This will follow from Theorem 3.1.

Several authors ([4], [5], [8], [10], [11], [12]) have considered nonsingular borderings  $A$  of  $B$  and investigated the properties of submatrices of  $A^{-1}$ . Under specific conditions on the bordering, one can recover a special  $g$ -inverse of  $B$  as a submatrix of  $A^{-1}$ . It turns out that in all such cases the condition  $\psi(B) = \psi(A)$  holds. The main theme of the present paper is to investigate borderings  $A$  of  $B$ , where  $A$  might be singular, or even rectangular. We show that if  $\psi(A) = \psi(B)$  is satisfied then many properties enjoyed by nonsingular borderings have analogs for singular (or rectangular) borderings as well. For example, any  $g$ -inverse of  $B$  can be obtained as a submatrix of  $A^-$  where  $A$  is a bordering of  $B$  with  $\psi(A) = \psi(B)$ . This will be shown in Section 4. In Section 5 we show how to obtain the Moore-Penrose inverse and the group inverse by a general, not necessarily nonsingular, bordering. In the next two sections we consider general borderings  $A$  of  $B$  and obtain some results concerning  $A^-$ .

We say that rank additivity holds in the matrix equation  $A = A_1 + \cdots + A_k$  if  $\text{rank}(A) = \text{rank}(A_1) + \cdots + \text{rank}(A_k)$ . Let  $R(A)$  and  $N(A)$  denote the range space of  $A$  and the null space of  $A$  respectively. We will need the following well-known result.

LEMMA 1.2. [2] *Let  $A, B$  be  $m \times n$  matrices. Then the following conditions are equivalent:*

- (i)  $\text{rank}(B) = \text{rank}(A) + \text{rank}(B - A)$ .
- (ii) Every  $B^-$  is a  $g$ -inverse of  $A$ .
- (iii)  $AB^-(B - A) = O$ ,  $(B - A)B^-A = O$  for any  $B^-$ .
- (iv) There exists a  $B^-$  that is a  $g$ -inverse of both  $A$  and  $B - A$ .

It follows from the proof of Lemma 1.1 that if  $\psi(B) = \psi(A)$  then rank additivity holds in

$$(1.3) \quad \begin{pmatrix} B & C \\ D & X \end{pmatrix} = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & X \end{pmatrix}$$

and in

$$(1.4) \quad \begin{pmatrix} B & C \\ D & X \end{pmatrix} = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & X \end{pmatrix}.$$

In Section 2 we discuss necessary and sufficient conditions for the block matrix  $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  to be a  $g$ -inverse of  $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$  under the assumption of rank additivity in (1.3) and (1.4). In section 3, necessary and sufficient conditions for the block matrix  $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  to be a  $g$ -inverse of  $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$  are considered under the assumption  $\psi(A) = \psi(B)$ . Certain related results are also proved. Some additional references on  $g$ -inverses of bordered matrices as well as generalizations of Cramer's rule are [1], [14], [16], [17].

**2. G-inverses of a bordered matrix .** Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  be a block matrix which is a bordering of  $B$ . In this section we will study some necessary and sufficient

conditions for a partitioned matrix  $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$ , conformal with  $A^*$ , to be a  $g$ -inverse of  $A$ .

**THEOREM 2.1.** *Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ . Then rank additivity holds in (1.3) and (1.4) and  $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  is a  $g$ -inverse of  $A$  if and only if the following conditions hold.*

- (i)  $BEB = B, CGC = C, DFD = D, XGC = DFX = -DEC, X = XYX - DEC$ .
  - (ii)  $CYD, BFX, CYX, XGB, XYD, BEC, DEB, CGB, BFD$  are null matrices.
- Furthermore, if  $EBE = E$ , then  $X = XYX$ .

*Proof.* "Only if" part: Assume rank additivity in (1.3) and (1.4) and that  $H$  is a  $g$ -inverse of  $A$ . Then by (ii) of Lemma 1.2,  $H$  is also a  $g$ -inverse of each summand matrix in (1.3) and (1.4). Using the definition of  $g$ -inverse, we easily get  $BEB = B, CGC = C, DFD = D, XYD = O, CYX = O$ , and

$$(2.1) \quad DFX + XYX = X, \quad XGC + XYX = X.$$

On the other hand, by (iii) of Lemma 1.2, we have

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & C \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow BEC = O,$$

$$\begin{pmatrix} O & C \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & O \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CGB = O,$$

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O \\ D & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow BFD = O, \quad BFX = O,$$

$$\begin{pmatrix} O & C \\ O & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O \\ D & X \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CYD = O, \quad CYX = O,$$

$$\begin{pmatrix} O & C \\ O & X \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & O \\ D & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix} \Rightarrow CYD = O, \quad XYD = O.$$

$$\left. \begin{aligned} \begin{pmatrix} O & O \\ D & X \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & O \\ O & O \end{pmatrix} &= \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & C \\ O & X \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & O \\ O & O \end{pmatrix} &= \begin{pmatrix} O & O \\ O & O \end{pmatrix} \end{aligned} \right\} \Rightarrow XGB = O, \quad DEB = O,$$

$$(2.2) \quad \left. \begin{aligned} \begin{pmatrix} O & O \\ D & X \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & C \\ O & O \end{pmatrix} &= \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ D & O \end{pmatrix} \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \begin{pmatrix} O & C \\ O & X \end{pmatrix} &= \begin{pmatrix} O & O \\ O & O \end{pmatrix} \end{aligned} \right\} \Rightarrow XGC = DFX = -DEC.$$

Also, (2.1) and (2.2) imply  $X = XYX - DEC$ .

“If” part: If the conditions (i) and (ii) hold, then it is easy to verify that  $H$  is a  $g$ -inverse of each summand matrix in (1.3) and (1.4). By (iv) in Lemma 1.2, rank additivity holds in (1.3) and (1.4). It is also easily verified that  $H$  is a  $g$ -inverse of  $A$ .

If  $EBE = E$ , then  $DEC = O$  and so  $X = XYX$ .  $\square$

We note certain consequences of Theorem 2.1.

COROLLARY 2.2. Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ . Then rank additivity holds in (1.3) and (1.4) and the matrix  $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$  is a  $g$ -inverse of  $A$  if and only if the following conditions hold.

(i)  $BEB = B, CGC = C, DFD = D, DEC = -X$ .

(ii)  $BEC, DEB, CGB, BFD$  are null matrices.

Furthermore if  $EBE = E$ , then  $X = O$ .

COROLLARY 2.3. Let  $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$ . Then  $R(B) \cap R(C) = \{0\}$ ,  $R(B^*) \cap R(D^*) = \{0\}$  and  $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  is a  $g$ -inverse of  $A$  if and only if the following conditions hold.

(i)  $BEB = B, CGC = C, DFD = D$ .

(ii)  $CYD, DEC, BEC, DEB, CGB, BFD$  are null matrices.

In this case, the  $g$ -inverses of  $A$  have the block independence property.

REMARK 2.4. As the conditions  $R(B) \cap R(C) = \{0\}$ ,  $R(B^*) \cap R(D^*) = \{0\}$  together with  $X = O$  imply rank additivity in (1.3) and (1.4), Corollary 2.3 is a direct consequence of Theorem 2.1. In particular, conditions (i) and (ii) indicate that the block matrices in  $\begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  can be independently chosen if it is a  $g$ -inverse of

$A$ . In other words, the  $g$ -inverses of  $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$  have the block independence property. Thus Corollary 2.3 complements the known result (see Theorem 3.1 in [15] and Lemma 5(1.2e) in [7]) that the  $g$ -inverses of  $A$  have the block independence property if and only if

$$\begin{aligned} \text{rank}(A) &= \text{rank} \begin{pmatrix} B \\ D \end{pmatrix} + \text{rank}(C) \\ &= \text{rank} \begin{pmatrix} B & C \end{pmatrix} + \text{rank}(D). \end{aligned}$$

The next result can also be viewed as a generalization of Corollary 2.3. This type of rank additivity has been considered, for example, in [13].

THEOREM 2.5. Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  and suppose

$$\text{rank}(A) = \text{rank}(B) + \text{rank}(C) + \text{rank}(D) + \text{rank}(X).$$

Then  $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  is a  $g$ -inverse of  $A$  if and only if the following conditions hold.

- (i)  $BEB = B$ ,  $CGC = C$ ,  $DFD = D$ ,  $XYX = X$ .
- (ii)  $BFX$ ,  $CYD$ ,  $CYX$ ,  $DFX$ ,  $XGB$ ,  $XGC$ ,  $XYD$ ,  $BEC$ ,  $BFD$ ,  $CGB$ ,  $DEB$ ,  $DEC$  are null matrices.

*Proof.* Note that the condition  $\text{rank}(A) = \text{rank}(B) + \text{rank}(C) + \text{rank}(D) + \text{rank}(X)$  implies rank additivity in

$$A = \begin{pmatrix} B & O \\ O & O \end{pmatrix} + \begin{pmatrix} O & C \\ O & O \end{pmatrix} + \begin{pmatrix} O & O \\ D & O \end{pmatrix} + \begin{pmatrix} O & O \\ O & X \end{pmatrix}.$$

Now the proof is similar to that of Theorem 2.1.  $\square$

A generalization of Theorem 2.5 is stated next; the proof is omitted.

**THEOREM 2.6.** Let  $A = (A_{i,j})$ ,  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$  be an  $m \times n$  block matrix. If  $\text{rank}(A) = \sum_{i=1}^m \sum_{j=1}^n \text{rank}(A_{i,j})$ , then  $G = (G_{l,s})$ ,  $l = 1, 2, \dots, n$ ,  $s = 1, 2, \dots, m$  is a  $g$ -inverse of  $A$  if and only if the following equations hold.

$$A_{i,j}G_{j,l}A_{l,s} = \begin{cases} A_{i,j} & (i,j) = (l,s) \\ O & (i,j) \neq (l,s) \end{cases}.$$

**3. G-inverses of a block matrix  $A$  with  $\psi(A) = \psi(B)$ .** Let  $A$  and  $H$  be matrices of order  $m \times n$  and  $n \times m$  respectively, partitioned as follows:

$$(3.1) \quad A = \begin{matrix} & \begin{matrix} q_1 & q_2 \end{matrix} \\ \begin{matrix} p_1 \\ p_2 \end{matrix} & \begin{pmatrix} B & C \\ D & X \end{pmatrix} \end{matrix} \quad \text{and} \quad H = \begin{matrix} \begin{matrix} p_1 & p_2 \end{matrix} \\ \begin{matrix} q_1 \\ q_2 \end{matrix} & \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \end{matrix},$$

where  $p_1 + p_2 = m$  and  $q_1 + q_2 = n$ . By  $\eta(A)$  we denote the row nullity of  $A$ , which by definition is the number of rows minus the rank of  $A$ . If  $m = n$ ,  $A$  is nonsingular,  $H = A^{-1}$  and if  $A$  and  $H$  are partitioned as in (3.1) then it was proved by Fiedler and Markham [10], and independently by Gustafson [9], that

$$(3.2) \quad \eta(B) = \eta(Y).$$

The following result, proved in [3], will be used in the sequel. We include an alternative simple proof for completeness.

**LEMMA 3.1.** Let  $A$  and  $H$  be matrices of order  $m \times n$  and  $n \times m$  respectively, partitioned as in (3.1). Assume  $\text{rank}(A) = r$  and  $\text{rank}(H) = k$ . Then the following assertions are true.

- (i) If  $AHA = A$ , then

$$-(m - r) \leq \eta(Y) - \eta(B) \leq n - r.$$

- (ii) If  $HAH = H$ , then

$$-(n - k) \leq \eta(B) - \eta(Y) \leq m - k.$$

*Proof.* (i) According to a result on bordered matrices and g-inverses [11, Theorem 1], there exist matrices  $P, Q$  and  $Z$  of order  $m \times (m-r)$ ,  $(n-r) \times n$  and  $(n-r) \times (m-r)$  respectively, such that the matrix

$$S = \begin{pmatrix} A & P \\ Q & Z \end{pmatrix}$$

is nonsingular and the submatrix formed by the first  $n$  rows and the first  $m$  columns of  $T = S^{-1}$  is  $W$ . Thus we may write

$$S = \begin{matrix} & q_1 & q_2 & m-r \\ \begin{matrix} p_1 \\ p_2 \\ n-r \end{matrix} & \begin{pmatrix} B & C & P_1 \\ D & X & P_2 \\ Q_1 & Q_2 & Z \end{pmatrix} \end{matrix} \quad \text{and} \quad T = \begin{matrix} & p_1 & p_2 & n-r \\ \begin{matrix} q_1 \\ q_2 \\ m-r \end{matrix} & \begin{pmatrix} E & F & U_1 \\ G & Y & U_2 \\ V_1 & V_2 & W \end{pmatrix} \end{matrix}.$$

Since  $S$  is nonsingular, we have, using (3.2),

$$\eta(B) = \eta\left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right) = q_2 + m - r - \text{rank}\left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right).$$

Now by Lemma 1.1

$$\text{rank}(Y) \leq \text{rank}\left(\begin{pmatrix} Y & U_2 \\ V_2 & W \end{pmatrix}\right) \leq \text{rank}(Y) + m + n - 2r,$$

and hence

$$-(m-r) \leq \eta(Y) - \eta(B) \leq n-r.$$

The result (ii) follows from (i).  $\square$

The following result, proved using Lemma 3.1, will be used in the sequel.

**THEOREM 3.2.** Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  with  $\psi(A) = \psi(B)$ . Then for any g-inverse

$$A^- = \begin{pmatrix} E & F \\ G & Y \end{pmatrix} \text{ of } A, Y = O.$$

*Proof.* Assume the sizes of the block matrices in  $A$  to be as in (3.1). By Lemma 3.1 we have

$$-(m-r) \leq \eta(Y) - \eta(B) \leq n-r.$$

It follows that

$$-m+r \leq q_2 - \text{rank}(Y) - p_1 + \text{rank}(B).$$

Using  $\psi(A) = \psi(B)$  and the inequality above,  $\text{rank}(Y) = 0$  and hence  $Y = O$ .  $\square$

**THEOREM 3.3.** Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ . Then  $\psi(A) = \psi(B)$  and  $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  is a g-inverse of  $A$  if and only if the following equations hold.

- (i)  $Y = O$ ,  $BEB = B$ ,  $GC = I$ ,  $DF = I$ .
  - (ii)  $DEC = -X$ .
  - (iii)  $BEC = O$ ,  $DEB = O$ ,  $BF = O$ ,  $GB = O$ .
- Furthermore, if  $EBE = E$ , then  $X = O$ .

*Proof.* If  $H$  is a  $g$ -inverse of  $A$  with  $\psi(A) = \psi(B)$ , then by Theorem 3.2, we know  $Y = O$ . From the proof of Lemma 1.1, the condition  $\psi(A) = \psi(B)$  also indicates rank additivity in (1.3) and (1.4). Note that  $C$  and  $D$  are also of full column rank and of full row rank respectively under the condition  $\psi(A) = \psi(B)$ . Then the proof of the theorem is similar to that of Theorem 2.1.  $\square$

The proof of the following result is also similar and is omitted.

**THEOREM 3.4.** Let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ ,  $H = \begin{pmatrix} E & F \\ G & Y \end{pmatrix}$  and consider the statements:

- (i)  $Y = O$ ,  $BEB = B$ ,  $GC = I$ ,  $DF = I$ ,  $BF = O$ ,  $GB = O$ .
- (ii)  $EB + FD$  is hermitian.
- (iii)  $BE + CG$  is hermitian.
- (iv)  $EBE + FDE = E$  (v)  $EBE + ECG = E$ .

Then

- (a)  $\psi(A) = \psi(B)$  and  $H \in A\{1, 2, 3\}$  if and only if (i), (ii), (iv) hold,  $DEC = -X$ ,  $EC = FDEC$  and  $DEB = O$ .
- (b)  $\psi(A) = \psi(B)$  and  $H \in A\{1, 2, 4\}$  if and only if (i), (iii), (v) hold,  $DEC = -X$ ,  $DE = DECG$  and  $BEC = O$ .
- (c)  $\psi(A) = \psi(B)$  and  $H = A^\dagger$  if and only if (i)-(v) hold,  $DE + XG = O$  and  $EC + FX = O$ .

The two previous results will be used in the proof of the next result.

**THEOREM 3.5.** let  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ . Then the following conditions are equivalent:

- (1)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$ .
- (2)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$  is a  $g$ -inverse of  $A$ .
- (3)  $X = O$ ,  $C^\dagger C = I$ ,  $DD^\dagger = I$ ,  $BD^\dagger = O$ ,  $C^\dagger B = O$ .
- (4)  $X = O$ ,  $C^\dagger C = I$ ,  $DD^\dagger = I$ ,  $BD^* = O$ ,  $C^* B = O$ .
- (5)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$  is a  $g$ -inverse of  $A$  for some  $E \in B^{\{1,2\}}$ .
- (6)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$  is a  $g$ -inverse of  $A$  for some  $E$ .
- (7)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} B & C \\ D & X \end{pmatrix}^\dagger = \begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}$  for some matrices  $F, G, Y$ .
- (8)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} B^\dagger & F \\ C^\dagger & Y \end{pmatrix}$  is a  $\{1, 2, 3\}$ -inverse of  $A$  for some  $F, Y$ .
- (9)  $\psi(A) = \psi(B)$  and  $\begin{pmatrix} B^\dagger & D^\dagger \\ G & Y \end{pmatrix}$  is a  $\{1, 2, 4\}$ -inverse of  $A$  for some  $G, Y$ .

*Proof.* Clearly, (1)  $\Rightarrow$  (2).

(2)  $\Rightarrow$  (3): This follows from Theorem 3.3.



(3)  $\Leftrightarrow$  (4): Since  $BD^\dagger = O$  and  $C^\dagger B = O$  are equivalent to  $BD^* = O$  and  $C^*B = O$  respectively, we have this implication.

(3)  $\Rightarrow$  (1): Note that  $BD^\dagger = O$  and  $C^\dagger B = O$  imply  $DB^\dagger = O$  and  $B^\dagger C = O$ . Then it is easy to verify that  $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$  is  $A^\dagger$  thus (1) holds.

Clearly, (1)  $\Rightarrow$  (5)  $\Rightarrow$  (6).

(6)  $\Rightarrow$  (3): By Theorem 3.3, if  $\begin{pmatrix} E & D^\dagger \\ C^\dagger & X^\dagger \end{pmatrix}$  is a  $g$ -inverse of  $A$  for some matrix  $E$ , then we have  $X^\dagger = O$ ,  $C^\dagger C = I$ ,  $DD^\dagger = I$ ,  $BD^\dagger = O$  and  $C^\dagger B = O$ . Note that  $X^\dagger = O \Leftrightarrow X = O$ , thus (3) holds.

(6)  $\Rightarrow$  (1): This follows from (6)  $\Rightarrow$  (3) and (3)  $\Rightarrow$  (1).

Obviously, (1)  $\Rightarrow$  (7), (1)  $\Rightarrow$  (8) and (1)  $\Rightarrow$  (9).

(7)  $\Rightarrow$  (1): By Theorem 3.3, we have  $X = O$ ,  $Y = O$ ,  $GC = I$ ,  $DF = I$ ,  $BF = O$  and  $GB = O$ . Clearly,  $G \in C\{1, 2, 4\}$  and  $F \in D\{1, 2, 3\}$ . Using the hermitian property of the matrices  $\begin{pmatrix} B & C \\ D & X \end{pmatrix}$ ,  $\begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix}$ ,  $\begin{pmatrix} B^\dagger & F \\ G & Y \end{pmatrix} \begin{pmatrix} B & C \\ D & X \end{pmatrix}$ ,  $BB^\dagger$  and  $B^\dagger B$ , it is easy to conclude that  $CG$  and  $FD$  are also hermitian. Thus  $F = D^\dagger$  and  $G = C^\dagger$ . Note that  $Y = X^\dagger = O$  and (1) is proved.

Similarly, using Theorem 3.4 we can show (8)  $\Rightarrow$  (1) and (9)  $\Rightarrow$  (1) and the proof is complete.  $\square$

#### 4. Obtaining any $g$ -inverse by bordering.

By Theorem 3.3 if  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$

with  $\psi(A) = \psi(B)$  and if  $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$  is a  $g$ -inverse of  $A$ , then  $E$  is a  $g$ -inverse of  $B$  which also satisfies  $DEC = -X$ ,  $BEC = O$  and  $DEB = O$ . Such an  $E$ , hereafter, will be denoted by  $E_{(C,D,X)}$ . Note that  $E_{(C,D,X)}$  is not uniquely determined by  $C, D, X$ , since  $A^-$  is not unique. In this section we will investigate the converse problem, that is: for a given  $g$ -inverse  $E$  of  $B$ , how to construct  $C, D$  and  $X$  so that  $H = \begin{pmatrix} E & F \\ G & O \end{pmatrix}$

is a  $g$ -inverse of  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  with  $\psi(A) = \psi(B)$  for some matrices of proper sizes.

We first state some well-known lemmas to be used later; see, for example, [4], [6].

LEMMA 4.1. *The following three statements are equivalent: (i)  $E$  is a  $g$ -inverse of  $B$ , (ii)  $BE$  is an idempotent matrix and  $\text{rank}(BE) = \text{rank}(B)$ , and (iii)  $EB$  is an idempotent matrix and  $\text{rank}(EB) = \text{rank}(B)$ .*

LEMMA 4.2.  *$E$  is a  $\{1, 2\}$ -inverse of  $B$  if and only if  $E$  is a  $g$ -inverse of  $B$  and  $\text{rank}(E) = \text{rank}(B)$ .*

LEMMA 4.3. *Let  $H = UV$  be a rank factorization of a square matrix. Then the following three statements are equivalent: (i)  $H$  is an idempotent matrix, (ii)  $I - H$  is an idempotent matrix, and (iii)  $VU = I$ .*

THEOREM 4.4. (i) *Let  $E$  be a  $g$ -inverse of the  $p_1 \times q_1$  matrix  $B$  with  $\text{rank}(B) = r$ . Then there exist  $C, D$ , and  $X$  such that  $E = E_{(C,D,X)}$ , where  $\text{rank}(C) \leq p_1 - r$  and  $\text{rank}(D) \leq q_1 - r$ .*

(ii) If  $E = E_{(C,D,X)}$ , then there exist matrices  $U, V, \bar{U}$  and  $\bar{V}$  such that

$$I - BE = (C \ U) \begin{pmatrix} G \\ V \end{pmatrix} \text{ and } I - EB = (F \ \bar{U}) \begin{pmatrix} D \\ \bar{V} \end{pmatrix}$$

are the rank factorizations of  $I - BE$  and  $I - EB$  respectively.

(iii)  $\text{rank}(E_{(C,D,X)}) = \text{rank}(B) + \text{rank}(R)$ , where

$$(4.1) \quad R = \begin{pmatrix} -X & DEU \\ \bar{V}EC & \bar{V}EU \end{pmatrix}$$

for some matrices  $U$  and  $\bar{V}$  as in (ii).

*Proof.* For a given g-inverse  $E$  of  $B$ , we use rank factorizations of  $I - BE$  and  $I - EB$ , by which there exist  $C, D, X, F, G, U, \bar{U}, V,$  and  $\bar{V}$  satisfying the following identities

$$(4.2) \quad I - BE = (C \ U) \begin{pmatrix} G \\ V \end{pmatrix},$$

$$(4.3) \quad I - EB = (F \ \bar{U}) \begin{pmatrix} D \\ \bar{V} \end{pmatrix},$$

$$X = -DEC.$$

To prove (i), we only need to show that these  $C, D, X, F$  and  $G$  along with  $Y = O$  satisfy the conditions (i),(ii) and (iii) in Theorem 3.3. In fact, from (4.2) and (4.3), we have, in view of Lemma 4.3, that  $\begin{pmatrix} G \\ V \end{pmatrix} (C \ U) = I$  and  $\begin{pmatrix} D \\ \bar{V} \end{pmatrix} (F \ \bar{U}) = I$ , implying

$$GC = I \text{ and } DF = I.$$

Again from (4.2) and (4.3), we have, by  $(I - BE)B = O$  and  $B(I - EB) = O$ ,

$$(4.4) \quad \begin{pmatrix} G \\ V \end{pmatrix} B = O \text{ and } B(F \ \bar{U}) = O,$$

and by  $BE(I - BE) = O$  and  $(I - EB)EB = O$ ,

$$(4.5) \quad BE(C \ U) = O \text{ and } \begin{pmatrix} D \\ \bar{V} \end{pmatrix} EB = O.$$

Now by (4.4),  $GB = O$  and  $BF = O$ , and by (4.5),  $BEC = O$  and  $DEB = O$ .

(ii) Let  $E = E_{(C,D,X)}$ . By Theorem 3.3,  $BEC = O$ , which means  $R(C) \subseteq N(BE) = R(I - BE)$ . Note that  $C$  is of full column rank under the condition  $\psi(A) = \psi(B)$ . Thus there exists a matrix  $U$  so that  $R((C \ U)) = R(I - BE)$

and the matrix  $(C \ U)$  is of full column rank. Hence, there exists a matrix of full row rank which can be partitioned as  $\begin{pmatrix} G \\ V \end{pmatrix}$  such that

$$I - BE = (C \ U) \begin{pmatrix} G \\ V \end{pmatrix}.$$

On the other hand,  $DEB = O$  implies  $N(I - EB) = R(EB) \subseteq N(D)$ . So there exists a matrix  $\bar{V}$  such that  $\begin{pmatrix} D \\ \bar{V} \end{pmatrix}$  is of full row rank and

$$N(I - EB) = N\left(\begin{pmatrix} D \\ \bar{V} \end{pmatrix}\right).$$

From this we conclude that there exists a matrix of full column rank which can be partitioned as  $(F \ \bar{U})$  such that

$$I - EB = (F \ \bar{U}) \begin{pmatrix} D \\ \bar{V} \end{pmatrix}.$$

Now we prove (iii). If  $E = E_{(C,D,X)}$ , then from the proof of (ii) there exist matrices  $U, V, \bar{U}$  and  $\bar{V}$  such that (4.2) and (4.3) hold. Hence  $BE(C \ U) = O$  and  $\begin{pmatrix} D \\ \bar{V} \end{pmatrix} EB = O$ . Therefore we have

$$\begin{aligned} \begin{pmatrix} B \\ D \\ \bar{V} \end{pmatrix} E (B \ C \ U) &= \begin{pmatrix} B & O & O \\ O & DEC & DEU \\ O & \bar{V}EC & \bar{V}EU \end{pmatrix} \\ &= \begin{pmatrix} B & O \\ O & R \end{pmatrix}, \end{aligned}$$

where  $R = \begin{pmatrix} D \\ \bar{V} \end{pmatrix} E (C \ U)$ .

On the other hand,

$$\begin{aligned} (E \ F \ \bar{U}) \begin{pmatrix} B & O \\ O & R \end{pmatrix} \begin{pmatrix} E \\ G \\ V \end{pmatrix} &= EBE + (F \ \bar{U}) R \begin{pmatrix} G \\ V \end{pmatrix} \\ &= EBE + (I - EB)E(I - BE) \\ &= E. \end{aligned}$$

Thus we have  $\text{rank}(E_{(C,D,R)}) = \text{rank}(B) + \text{rank}(R)$ .  $\square$

Theorem 4.4(i) and its proof show that for a given matrix  $B$  and its  $g$ -inverse  $E$  we can find matrices  $C$  of full column rank with  $R(C) \subseteq N(BE)$  and  $D$  of full row rank with  $R(EB) \subseteq N(D)$ , as well as  $X = -DEC$ ,  $F$  and  $G$  such that matrix

$\begin{pmatrix} E & F \\ G & O \end{pmatrix}$  is a  $g$ -inverse of  $A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$  with  $\psi(A) = \psi(B)$ . Furthermore, we have the following.

**COROLLARY 4.5.** *Let  $B$  and its  $g$ -inverse  $E$  be given. Then the matrix  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  satisfies  $\psi(A) = \psi(B)$  and has a  $g$ -inverse of the form  $\begin{pmatrix} E & F \\ G & O \end{pmatrix}$  if and only if  $C$  is of full column rank with  $R(C) \subseteq N(BE)$  and  $D$  of full row rank with  $R(EB) \subseteq N(D)$ . In this case,  $X = -DEC$ ,  $F \in D\{1, 3\}$ ,  $G \in C\{1, 4\}$ ,  $BF = O$  and  $GB = O$ .*

*Proof.* Necessity: This follows from Theorem 3.3.

Sufficiency: The proof of sufficiency is similar to that of Theorem 4.4(i), (ii).  $\square$

As a special case we recover the following known result.

**COROLLARY 4.6.** [11, Theorem 1] *Let  $E$  be a  $g$ -inverse of  $B$ . Then for any matrix  $C$  of full column rank with  $R(C) = N(BE)$  and any matrix  $D$  of full row rank with  $N(D) = R(EB)$ , the matrix*

$$A = \begin{pmatrix} B & C \\ D & -DEC \end{pmatrix}$$

*is nonsingular and*

$$A^{-1} = \begin{pmatrix} E & F \\ G & O \end{pmatrix},$$

*where  $F \in D\{1, 3\}$ ,  $BF = O$ ,  $G \in C\{1, 4\}$  and  $GB = O$ .*

**5. Moore-Penrose inverse and group inverse by bordering.** For a given  $g$ -inverse  $E$  of  $B$ , Corollary 4.5 shows that  $C$  and  $D$  can be chosen with the conditions  $R(C) \subseteq N(BE)$  and  $R(D^*) \subseteq N((EB)^*)$  so that  $A = \begin{pmatrix} B & C \\ D & -DEC \end{pmatrix}$  satisfies  $\psi(A) = \psi(B)$  and has a  $g$ -inverse of the form  $\begin{pmatrix} E & F \\ G & O \end{pmatrix}$ . Further, Corollary 4.6 provides an approach to border the matrix  $B$  into a nonsingular matrix such that in its inverse, the block matrix on the upper left corner is  $E$ . We now show how to border the matrix if  $E$  is the Moore-Penrose inverse or the group inverse of  $B$ .

**THEOREM 5.1.** *Let  $B$  be given. Then the matrix  $A = \begin{pmatrix} B & C \\ D & X \end{pmatrix}$  satisfies  $\psi(A) = \psi(B)$  and has a  $g$ -inverse of the form  $\begin{pmatrix} B^\dagger & F \\ G & O \end{pmatrix}$  if and only if  $C$  has full column rank with  $R(C) \subseteq N(B^*)$  and  $D$  has full row rank with  $R(D^*) \subseteq N(B)$ . In this case,  $X = -DB^\dagger C = O$  and*

$$A^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}.$$

*Proof.* Note that  $N(BB^\dagger) = N(B^*)$  and  $N((EB)^*) = N(B^\dagger) = N(B)$ , and the necessity and sufficiency follow from Corollary 4.5.

It is easy to verify that  $\begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}$  is a  $g$ -inverse of  $A$ . Thus by Corollary 3.5(2), we have

$$A^\dagger = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix},$$

where  $X = -DB^\dagger C = O$ .  $\square$

Combining Corollary 4.6 with Theorem 5.1, we have

**COROLLARY 5.2.** [5] *Let  $B$  be a  $p_1 \times q_1$  matrix with  $\text{rank}(B) = r$ . Suppose the columns of  $C \in C_{p_1-r}^{p_1 \times (p_1-r)}$  are a basis of  $N(B^*)$  and the columns of  $D^* \in C_{q_1-r}^{q_1 \times (q_1-r)}$  are a basis for  $N(B)$ . Then the matrix*

$$A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$$

*is nonsingular and its inverse is*

$$A^{-1} = \begin{pmatrix} B^\dagger & D^\dagger \\ C^\dagger & O \end{pmatrix}.$$

If  $B$  is square and has group inverse, we can get a bordering  $\begin{pmatrix} B & * \\ * & O \end{pmatrix}$  of  $B$  such that it has a  $g$ -inverse in the form  $\begin{pmatrix} B^\# & * \\ * & O \end{pmatrix}$ . Part (ii) of the following result is known. We generalize it to any bordering, not necessarily nonsingular, in part (i).

**THEOREM 5.3.** *Let  $B$  be  $n \times n$  and with index 1. Then*

(i) *there exist matrices  $C$  of full column rank with  $R(C) \subseteq N(B)$  and  $D$  of full row rank with  $R(B) \subseteq N(D)$  which satisfy  $DC = I$  such that  $\begin{pmatrix} B^\# & C \\ D & O \end{pmatrix}$  is a  $g$ -inverse of  $\begin{pmatrix} B & C \\ D & O \end{pmatrix}$  with  $\psi(A) = \psi(B)$ ;*

(ii) ([8], [14], [17]) *for any matrix  $C$  of full column rank with  $R(C) = N(B)$  and any matrix  $D$  of full row rank with  $R(B) = N(D)$ , the matrix*

$$A = \begin{pmatrix} B & C \\ D & O \end{pmatrix}$$

*is nonsingular and*

$$A^{-1} = \begin{pmatrix} B^\# & C(DC)^{-1} \\ (DC)^{-1}D & O \end{pmatrix}.$$

*Proof.* (i): Consider the rank factorization of  $I - BB^\#$  given by

$$I - BB^\# = (C \ U) \begin{pmatrix} D \\ V \end{pmatrix}.$$

Note that  $BB^\sharp = B^\sharp B$ , and we have

$$I - B^\sharp B = \begin{pmatrix} C & U \end{pmatrix} \begin{pmatrix} D \\ V \end{pmatrix}.$$

Obviously  $R(C) \subseteq N(A)$  and  $R(B) \subseteq N(D)$ . As in the proof of Theorem 4.4(i), we conclude that  $\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}$  is a  $g$ -inverse of  $\begin{pmatrix} B & C \\ D & O \end{pmatrix}$  with  $\psi(A) = \psi(B)$ , since  $X = -DB^\sharp C = O$ .

(ii): By Corollary 4.6, the nonsingularity of the matrix  $\begin{pmatrix} B^\sharp & C \\ D & O \end{pmatrix}$  under the conditions  $R(C) = N(A)$  and  $R(B) = N(D)$  can be easily seen. We now prove that for any matrix  $C$  of full column rank with  $R(C) = N(B)$  and any matrix  $D$  of full row rank with  $R(B) = N(D)$ ,  $DC$  is nonsingular.

In fact, if  $DCx = O$ , then  $Cx \in R(C)$  and  $Cx \in N(D)$ . Since  $R(C) = N(B)$ ,  $N(D) = R(B)$  and  $R(B) \cap N(B) = \{0\}$ , we have  $Cx = O$  and therefore  $x = 0$ . Thus  $DC$  is nonsingular.

By Lemma 4.3,  $C(DC)^{-1}D$  is an idempotent matrix and

$$I - BB^\sharp = I - B^\sharp B = C(DC)^{-1}D$$

is a rank factorization. From Corollary 4.6, we know that

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B & C \\ (DC)^{-1}D & O \end{pmatrix}$$

are nonsingular and in fact

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix}^{-1} = \begin{pmatrix} B^\sharp & C(DC)^{-1} \\ D & O \end{pmatrix}.$$

Note that

$$\begin{pmatrix} B & C(DC)^{-1} \\ D & O \end{pmatrix} = \begin{pmatrix} B & C \\ D & O \end{pmatrix} \begin{pmatrix} I & O \\ O & (DC)^{-1} \end{pmatrix}.$$

The result follows immediately from the two equations preceding the one above.  $\square$

REMARK 5.4. Theorem 5.3(ii) can be used to compute the group inverse of the matrix  $(I - T)^\sharp$  which plays an important role in the theory of Markov chains, where  $T$  is the transition matrix of a finite Markov chain. For an  $n$ -state ergodic chain, it is well-known that the transition matrix  $T$  is irreducible and that  $\text{rank}(I - T) = n - 1$  [6, Theorem 8.2.1]. Hence by Theorem 5.3(ii) we can compute the group inverse  $(I - T)^\sharp$  of  $I - T$  by a bordered matrix.

Let  $c$  be a right eigenvector of  $T$  and  $d$  a left eigenvector, that is  $c$  and  $d$  satisfy  $Tc = c$  and  $d^*T = d^*$ , respectively. Then the bordered matrix  $\begin{pmatrix} I - T & c \\ d^* & 0 \end{pmatrix}$  is nonsingular and

$$\begin{pmatrix} I - T & c \\ d^* & 0 \end{pmatrix}^{-1} = \begin{pmatrix} (I - T)^\sharp & \frac{c}{d^*c} \\ \frac{d}{d^*c} & o \end{pmatrix}.$$

Thus the group inverse  $(I - T)^\sharp$  can be obtained by computing the inverse of a nonsingular matrix.

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