SINGULAR VALUE DECOMPOSITION AND THE
MOORE–PENROSE INVERSE OF BORDERED MATRICES*

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Abstract. The singular value decomposition of a matrix is used to derive systematically the
Moore–Penrose inverse for a matrix bordered by a row and a column, in addition to the Moore–
Penrose inverse for the associated principal Schur complements.

1. Introduction. One of the most powerful methods of investigating the
Moore–Penrose inverse of a matrix [10] is through the singular value decomposi-
tion theory [8, p. 338]. In this paper we shall use this theory to obtain the
Moore–Penrose inverse, for the complex bordered matrix

\[ M = \begin{bmatrix} A & c \\ b^* & d \end{bmatrix}, \]

and for the associated Schur complement \( A - cd^\dagger b^* \), where \( b \) and \( c \) are columns and

\[ d^\dagger = \begin{cases} 0, & d = 0, \\ d^{-1}, & d \neq 0. \end{cases} \]

We shall show that both problems reduce to the question of finding the Moore–
Penrose inverse of certain partitioned block matrices, in which the individual
blocks are of a simple nature. This explains why several completely independent cases do occur, as were derived piece by piece in [1], [3], [6], [7], [11] and [13] in a
very random manner.

From the singular value decomposition theory we know that for any complex
\( m \times n \) matrix \( A \), of rank \( \rho(A) = r \), there exist unitary matrices \( U \) and \( V \) such that

\[ V^* AU = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ 0 & \cdots & \sigma_r \end{bmatrix}, \quad r = \rho(A), \]

and \( \sigma_i > 0 \) are the singular values of \( A \), i.e., the positive square roots of the positive
eigenvalues of \( A^* A \) and \( AA^* \).

This reduction is extremely useful since the Moore–Penrose inverse of \( A \) is
given by

\[ A^+ = U \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^*. \]

If we partition \( U \) and \( V \) as

\[ U = [U_1, U_2], \quad V = [V_1, V_2], \]

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with $U_1 = [u_1, \cdots, u_r]$ and $V_1 = [v_1, \cdots, v_r]$, then

$$A^*A u_i = \sigma_i^2 u_i, \quad i = 1, \cdots, r,$$

$$A u_i = 0, \quad i = r + 1, \cdots, n,$$

and

$$A A^* v_i = \sigma_i^2 v_i, \quad i = 1, \cdots, r,$$

$$A^* v_i = 0, \quad i = r + 1, \cdots, m.$$

Note that $U_2$ and $V_2$ contain orthonormal bases for $\ker (A)$ and $\ker (A^*)$, respectively, and that $U_1^* U_1 = I = V_1^* V_1$, $U_1 U_1^* = I - U_2 U_2^*$ and $V_1 V_1^* = I - V_2 V_2^*$. Using this, we may write

$$A = V_1 \Sigma U_1^* \quad \text{and} \quad A^\dagger = U_1 \Sigma^{-1} V_1^*,$$

from which several useful relations can be derived, for example,

$$U_1 U_1^* = A^\dagger A, \quad U_2 U_2^* = I - A^\dagger A, \quad \text{and} \quad V_1 V_1^* = A A^\dagger, \quad V_2 V_2^* = I - A A^\dagger.$$

Consider now the following transformation which diagonalizes the matrix $A$ in $M$:

$$\begin{bmatrix} V^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & c \\ b^* & d \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^* c \\ b^* U \end{bmatrix}.$$  

If we let $p^* = b^* U = [b^* U_1, b^* U_2] = [p_1^*, p_2^*]$ and

$$q = V^* c = [V_1^* c, V_2^* c] = [q_1, q_2],$$

then the problem of finding $M^\dagger$ has been reduced to finding the Moore–Penrose inverse of the matrix

$$\begin{bmatrix} \Sigma & 0 & q_1 \\ 0 & 0 & q_2 \\ p_1^* & p_2^* & d \end{bmatrix},$$

which on rearranging rows and columns shows that

$$M^\dagger = \begin{bmatrix} U_1 & U_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & I \\ 0 & 1 & 0 \end{bmatrix} N^\dagger \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 1 \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} V_1^* & 0 \\ 0 & 1 \\ V_2^* & 0 \end{bmatrix},$$

(5)

$$= \begin{bmatrix} U_1 & 0 & U_2 \end{bmatrix} N^\dagger \begin{bmatrix} V_1^* & 0 \\ 0 & 1 \\ V_2^* & 0 \end{bmatrix},$$

where

$$N = \begin{bmatrix} \Sigma & q_1 & 0 \\ p_1^* & d & p_2^* \\ 0 & q_2 & 0 \end{bmatrix} = \begin{bmatrix} N_1 & 0 \\ 0 & q_2 \\ 0 & q_2 \end{bmatrix}. \quad \text{(continued)}$$
Using row operations, we see that \( N \) is row equivalent to the matrix

\[
\begin{bmatrix}
\Sigma & \mathbf{q}_1 & 0 \\
0 & z & \mathbf{p}_2^* \\
0 & 0 & 0
\end{bmatrix},
\]

where \( z = d - \mathbf{p}_1^* \Sigma^{-1} \mathbf{q}_1 \),

from which we easily see that there are 5 cases that can occur, namely:

\[
\begin{align*}
\rho(M) &= r + 2 \iff \mathbf{p}_2 \neq 0 \quad \text{and} \quad \mathbf{q}_2 \neq 0, \\
\rho(M) &= r + 1 \iff \begin{cases} 
\mathbf{p}_2 = 0 \quad \text{and} \quad \mathbf{q}_2 \neq 0 \\
\text{or} \quad \mathbf{p}_2 \neq 0 \quad \text{and} \quad \mathbf{q}_2 = 0 \\
\text{or} \quad \mathbf{p}_2 = 0, \quad \mathbf{q}_2 = 0 \quad \text{and} \quad z \neq 0,
\end{cases}
\end{align*}
\]

(6)

If we denote the range of a matrix \( X \) by \( R(X) \), then this may be interpreted in terms of \( \mathbf{b}, \mathbf{c} \) using (4) and the facts that

\[
\begin{align*}
\mathbf{p}_2 = 0 &\iff \mathbf{b} \in R(\mathbf{A}^\diamond), \\
\mathbf{q}_2 = 0 &\iff \mathbf{c} \in R(\mathbf{A}), \\
\mathbf{p}_1 = 0 &\iff \mathbf{h} = \mathbf{A}^\diamond \mathbf{b} = 0, \\
\mathbf{q}_1 = 0 &\iff \mathbf{k} = \mathbf{A}^\diamond \mathbf{c} = 0.
\end{align*}
\]

(7)

We shall similarly denote the row spaces of a matrix \( X \) by \( RS(X) \). In addition to (6), we have

\[
(8) \quad z = d - \mathbf{p}_1^* \Sigma^{-1} \mathbf{q}_1 = d - \mathbf{b}^h U \Sigma^{-1} V^* \mathbf{c} = d - \mathbf{b}^* \mathbf{A}^\dagger \mathbf{c},
\]

which is one of the leading Schur complements [2]. Hence, in finding \( N^\dagger \), there are five basically different cases to be considered dictated by the relative values of \( \mathbf{p}_2, \mathbf{q}_2 \), and \( z \); see Table 1. We shall take up the 5 cases in the next section.

**Table 1**

| \( \mathbf{p}_2 \) | \( \mathbf{q}_2 \) | \( z = d - \mathbf{b}^* \mathbf{A}^\dagger \mathbf{c} \) | \( \mathbf{b} \in R(\mathbf{A}^\diamond) \) | \( \mathbf{c} \in R(\mathbf{A}) \) | \( \rho(M) \) | **Formula for** \( M^\dagger \)
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>yes</td>
<td>yes</td>
<td>( r + 1 )</td>
</tr>
<tr>
<td>Case 2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>yes</td>
<td>yes</td>
<td>( r )</td>
</tr>
<tr>
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<td>( \neq 0 )</td>
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<td>no</td>
<td>no</td>
<td>( r + 2 )</td>
</tr>
<tr>
<td>Case 4</td>
<td>( \neq 0 )</td>
<td>0</td>
<td>free</td>
<td>no</td>
<td>yes</td>
<td>( r + 1 )</td>
</tr>
<tr>
<td>Case 5</td>
<td>0</td>
<td>( \neq 0 )</td>
<td>free</td>
<td>yes</td>
<td>no</td>
<td>( r + 1 )</td>
</tr>
</tbody>
</table>

2. **Five subcases.** We shall derive the Moore–Penrose inverse for the first two cases and then show how the other three cases can be obtained from these two. It is also evident that Case 5 follows from Case 4 simply by considering \( M^* \) and then interchanging \( \mathbf{A}, \mathbf{b}, \mathbf{c}, \) and \( d \) with \( \mathbf{A}^*, \mathbf{c}, \mathbf{b} \) and \( d^* \), respectively.

For the first two cases, the problem has been reduced to finding the Moore–Penrose inverse of the matrix

\[
N_1 = \begin{bmatrix}
\Sigma & \mathbf{q}_1 \\
\mathbf{p}_1^* & d
\end{bmatrix},
\]

where \( \Sigma \) is nonsingular.
Case 1. When $z = d - p_i^* \Sigma^{-1} q_i \neq 0$, the matrix $N_i$ is nonsingular, and [8, p. 147]

$$N_i^{-1} = \begin{bmatrix} \Sigma^{-1} + \Sigma^{-1} q_i p_i^* \Sigma^{-1} / z & -\Sigma^{-1} q_i / z \\ -p_i^* \Sigma^{-1} / z & 1 / z \end{bmatrix}.$$  \hfill (9)  

Hence on using (5) and (3) and the definitions of $p_i$, $q_i$ we get

$$M^* = \begin{bmatrix} U_i \Sigma^{-1} V_i^* + U_i \Sigma^{-1} q_i p_i^* \Sigma^{-1} V_i^* / z & -U_i \Sigma^{-1} q_i / z \\ -p_i^* \Sigma^{-1} V_i^* / z & 1 / z \end{bmatrix}$$

$$= \begin{bmatrix} A^* + k h^* / z & -k / z \\ -h^* / z & 1 / z \end{bmatrix}.$$  \hfill (10)

This is similar to (9) with $(\cdot)^{-1}$ replaced by $(\cdot)^*$.  

Case 2. The matrix $N$ has $\rho(N_i) = \rho(\Sigma) = r$, so that the calculation of $N_i^*$ is well known [9], [14], [5]. Using rank factorization, we may write

$$N_i = \begin{bmatrix} \Sigma_i \end{bmatrix}^{-1} \Sigma_i, [\begin{bmatrix} \Sigma_i \end{bmatrix} \begin{bmatrix} q_i \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \Sigma[I, O],$$

where $P = p_i^* \Sigma^{-1}$ and $Q = \Sigma^{-1} q_i$, and hence

$$N_i^* = \begin{bmatrix} \Sigma_i \end{bmatrix} (\Sigma_i + q_i q_i^*)^{-1} \Sigma_i (\Sigma_i + q_i q_i^*)^{-1} [\begin{bmatrix} \Sigma_i \end{bmatrix} \begin{bmatrix} q_i \end{bmatrix} = \begin{bmatrix} I \\ Q_i \end{bmatrix} (I + Q_i Q_i^*)^{-1} \Sigma_i^{-1} (I + P_i P_i^*)^{-1} [\begin{bmatrix} I \\ P_i \end{bmatrix}.$$

This reduces to

$$N_i^* = \begin{bmatrix} R \\ Q^* R \\ Q^* P_i^* \end{bmatrix},$$

where $R = [I - Q_i Q_i^* (I + Q_i Q_i^*)]^{-1} [I - P_i P_i^* (I + P_i P_i^*)]^{-1}$. Using (5) and substituting $P = h_i^* A_i V_i = h_i^* V_i$, $Q = U_i^* A_i c = U_i^* k$, we obtain

$$M^* = \begin{bmatrix} U_i R V_i^* \\ Q^* R V_i^* \\ Q^* P_i^* \end{bmatrix}$$

$$= \begin{bmatrix} I \\ k_i^* \end{bmatrix} [1 - k k_i^* (I + k_i^* k_i)^{-1}] A_i^* [I - h h_i^* (1 + h_i^* h_i)^{-1}] [I, h].$$  \hfill (11)

For the remaining three cases, the matrix $N_i^*$ could be calculated in a similar fashion, using (11) and the well-known results for $[U, V]^*$ [12, ex. 23, p. 71]. These expressions, however, become quite complicated. Instead, we shall use the well known fact that for any matrix

$$M = \begin{bmatrix} A & C \\ B & D \end{bmatrix},$$
the matrix

\[
MM^* = \begin{bmatrix}
A_1 & C_1 \\
C_1^* & D_1
\end{bmatrix} = \begin{bmatrix}
AA^* + CC^* & AB^* + CD^* \\
BA^* + DC^* & BB^* + DD^*
\end{bmatrix}
\]

is positive semi-definite and thus always satisfies \( R(C_1) \subseteq R(A_1), \ RS(C_1^*) \subseteq RS(A_1) \). In particular, if \( M \) is bordered as in (5), then so is \( MM^* \) and hence \( MM^* \) always satisfies either Case 1 or Case 2. The Moore–Penrose inverse is then calculated from \( M^t = M^*(MM^*)^t \). Before we can apply either (10) or (12), we have to calculate the principal Schur complement \( Z \) for \( MM^* \). This is just as easily done in general, for if

\[
M = \begin{bmatrix}
A & C \\
B^* & D
\end{bmatrix}
\]

and \( z = D - B^*A^tC \), then

\[
Z = B^*B + D^*D - (B^*A^* + DC^*)(AA^* + CC^*)^t(AB^* + CD^*)
\]

(13)

\[
= [B^*, D](I - [A, C][A, C]) \begin{bmatrix}
B \\
D^*
\end{bmatrix},
\]

Whence

\[
Z = 0 \iff R\left( \begin{bmatrix}
B \\
D^*
\end{bmatrix} \right) \subseteq \ker(I - [A, C][A, C]) = R\left( \begin{bmatrix}
A^打听 \\
C^打听
\end{bmatrix} \right)
\]

\[
\iff \exists X \ni A^*X = B, C^*X = D^*
\]

\[
\iff R(B) \subseteq R(A^*) \quad \text{and} \quad \exists H \ni D^* = C^*\{A^*B + (I - A^*A)H\}
\]

\[
\iff R(B) \subseteq R(A^*) \quad \text{and} \quad \exists H \ni Z^* = C^*(I - AA^*)H
\]

\[
\iff R(B) \subseteq R(A^*) \quad \text{and} \quad \begin{cases} \text{either} & R(C) \not\subseteq R(A) \\ \text{or} & R(C) \subseteq R(A) \quad \text{and} \quad z = 0. \end{cases}
\]

We are now in a position to complete the table of Moore–Penrose inverses, but we must still distinguish Cases 3 and 4 since, by a result in [4],

(14) \[
[A, c]^t = \begin{bmatrix}
A^t - kw^* \\
w^*
\end{bmatrix}, \quad [A, c]^t[A, c] = \begin{bmatrix}
A^tA - kw^*A & k - kw^*c \\
w^*A^t & w^*c
\end{bmatrix},
\]

where \( w^* = u^t + (1 - u^tu)k^tA^t(1 + k^tk)^{-1} \) and \( u = (I - AA^*)c, k = A^*c \). Suppose now that \( b \not\in R(A^*) \), i.e., \( Z \neq 0 \). Then for the Cases 3 and 4, we have, by (10),

(15) \[
M^t = \begin{bmatrix}
A^* & b \\
C^* & d^*
\end{bmatrix}\begin{bmatrix}
A^t & A^tC_1^*A_1^*/Z & A_1^*/Z \\
C^* & A^t/Z & 1/Z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A^*A^t & (A^*A^tC_1 - b)C^tA_1^*/Z & (b - A^*A_1^tC_1)/Z \\
A^tC_1^*A_1^*/Z & A^tA_1^*/Z & (d^* - e^*A^tA_1^tC_1)/Z
\end{bmatrix}.
\]
We may simplify this on using (14) and the definition of $A_1$ in conjunction with the identity

$$Z = b^*(I - A^*A)b + (b^*k - d)(w^*A b + d^*(w^*c - 1))$$

$$= v^*v + z[w^*A b + d^*(w^*c - 1)],$$

to yield

$$M^t = \left[ A^+ - kw^* + (z^*A^*w - v)(h^* + zw^*)/Z \quad (v - z^*A^*w)/Z \right]$$

$$w^* + z^*(c^*w - 1)(h^* + zw^*)/Z \quad z^*(c^*w + 1)/Z \right],$$

in which only $w$ has to be substituted.

**Case 3.** $b \notin R(A^*)$, $c \notin R(A)$, $Z \neq 0$. In this case, $w^* = u^*$, so that $c^*w = 1$ and $A^*w = 0$, which yields $Z = v^*v$ and thus

$$M^t = \left[ A^+ - ku^+ - v^*h^+ - zw^+ u^+ \quad v^*v^+ \right].$$

**Case 4.** $b \notin R(A^*)$, $c \in R(A)$, $Z \neq 0$. Since $w^* = k^*A^+(1 + k^*k)^{-1}$, it is easily seen that $A^*w = k(1 + k^*k)^{-1}$ and $c^*w = k^*k(1 + k^*k)^{-1}$, so that $Z$ reduces to

$$Z_1 = v^*v + zz^*(1 + k^*k)^{-1} \quad \text{or} \quad v^*v/Z = 1 - zz^*(1 + k^*k)^{-1}/Z.$$

A final substitution in (16) yields

$$M^t = \left[ A^+ - \frac{kk^*A^+}{1 + k^*k} + \frac{1}{Z} \frac{z^*k}{1 + k^*k} - v \right] \frac{(h^* + zk^*A^+)}{1 + k^*k} \frac{1}{Z} \left( v - \frac{z^*k}{1 + k^*k} \right)$$

$$\frac{k^*A^+}{1 + k^*k} - \frac{z^*}{Z(1 + k^*k)} \left( h^* + zk^*A^+ \right) \frac{1}{Z(1 + k^*k)}.$$

Using (18), the (1, 1)- and (2, 1)-elements may be simplified to $Z^{-1}[A^+ - kk^*A^tv^*v + z^*kh^*(1 + k^*k) - zvk^*A^+(1 + k^*k)^{-1}]$ and $v^*v/Z - z^*h^*/Z(1 + k^*k)$, which agrees with the results of [6].

**Case 5.** $b \in R(A^*)$, $c \notin R(A)$, $z = 0$. We can use [12] to give $(MM^*)^*$; however, this becomes unnecessarily complicated. It is easier to consider

$$M^* = \left[ A^* \quad b \right]$$

and then use Case 4. Having found $M^*$, we obtain $M^t$ from $M^t = (M^*)^*$. Hence one only has to star (19) and interchange $A, b, c, d, u, v, h, k$ with
\(A^*, c, b, d^*, v, u, k, \) and \(h, \) respectively. In fact,

\[
(M')_{11} = A^\dagger - A^\dagger h^h(1 + h^h)^{-1} + Z_1^{-1}[z^* h^*(1 + h^h)^{-1} - u^*] \cdot [k^* + z A^\dagger h(1 + h^h)^{-1}],
\]

\[
(M')_{21} = Z_1^{-1}[u^* - z^* h^*(1 + h^h)^{-1}],
\]

\[
(M')_{12} = A^\dagger h(1 + h^h)^{-1} - z^* Z_1^{-1}(1 + h^h)^{-1}[k + A^\dagger h z(1 + h^h)^{-1}],
\]

\[
(M')_{22} = -z^* Z_1^{-1}(1 + h^h)^{-1},
\]

where \(Z_1 = u^*u + z^* z(1 + h^h)^{-1} = Z_1^*.\)

### 3. Schur complements of \(M.\) We shall now use the theory of singular values to calculate systematically the Moore–Penrose inverse of the Schur complement \(\zeta = A - cd^* b^*\) of the matrix \(M,\) in terms of the complementary Schur complement \(z = d - b^* A^* c\) and the matrices \(A, b,\) and \(c,\) thus verifying the results of [3] and [7]. Since the case \(d = 0\) is trivial, we may assume, without loss of generality, that \(d = 1.\) Again, several basically distinct cases appear quite naturally in this approach.

Let us diagonalize \(A\) as in (1) to obtain

\[
L = V^* \zeta U = \begin{bmatrix}
\Sigma & 0 \\
0 & 0
\end{bmatrix} - \begin{bmatrix}
V_1^* c & [b^* U_1, b^* U_2] \\
V_2^* c & [b^* U_1, b^* U_2]
\end{bmatrix}
\]

where, as before, \(p_i = U_i^* b\) and \(q_i = V_i^* c.\) Hence the problem of finding \(\zeta^+\) is equivalent to finding the Moore–Penrose inverse of the specially partitioned block matrix \(L,\ i.e.,\)

\[
\tilde{\zeta}^+ = [U_1, U_2] L^\dagger \begin{bmatrix}
V_1^* \\
V_2^*
\end{bmatrix}.
\]

In principle, we could write down the most general \(L^\dagger;\) however, this is rather complicated, and it is therefore again advantageous to consider special cases before rather than after \(g\)-inversion of \(L.\) There are five main cases to be considered; see Table 2. Again, Cases 3 and 4 follow immediately from Cases 1 and 2 on considering \(\zeta^*\) and interchanging \(A, b, c\) with \(A^*, c, b,\) respectively.

<table>
<thead>
<tr>
<th>(p_2)</th>
<th>(q_2)</th>
<th>(b \in R(A^*))</th>
<th>(c \in R(A))</th>
<th>(z = d - b^* A^* c)</th>
<th>((A - cd^* b^*)^\dagger)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>yes</td>
<td>(\neq 0)</td>
</tr>
<tr>
<td>Case 2</td>
<td>—</td>
<td>0</td>
<td>—</td>
<td>yes</td>
<td>=0</td>
</tr>
<tr>
<td>Case 3</td>
<td>0</td>
<td>—</td>
<td>yes</td>
<td>—</td>
<td>(\neq 0)</td>
</tr>
<tr>
<td>Case 4</td>
<td>0</td>
<td>—</td>
<td>yes</td>
<td>—</td>
<td>=0</td>
</tr>
<tr>
<td>Case 5</td>
<td>(\neq 0)</td>
<td>(\neq 0)</td>
<td>no</td>
<td>no</td>
<td>—</td>
</tr>
</tbody>
</table>
When \( c \in R(A) \), i.e., \( q_2 = 0 \), we have

\[
L = \begin{bmatrix}
\Sigma - q_1 p_1^* & -q_1 p_2^* \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
S & T \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
G & 0 \\
H & 0
\end{bmatrix}^\dagger,
\]

so that

\[
\xi^\dagger = U_1 G \Sigma V_1^* + U_2 H V_1^*.
\]

Let us now evaluate \( G, H \) for the various cases using a variety of techniques.

Case 1. \( q_2 = 0, c \in R(A), z \neq 0 \). In this case, \( S \) is invertible since \( |S| = z |\Sigma| \). In fact [8, pp. 148, 225], it is easily seen that

\[
(22) \quad \Delta = \frac{U_1 G V_1^* + U_2 H V_1^*}{z},
\]

Thus, if \( P_2 \) were zero, this would give \( L \). In the general case, we may for simplicity factor \( L = S^{1/2} T = S^{1/2} I, S^{1/2} T \), so that \( L = S^{1/2} T S^{1/2} = S^{1/2} T S^{1/2} \) since both matrices are of full row rank.

We proceed by calculating \( S^{-1} T = -\Sigma^{-1} q_1 p_2^*/z \), which may be derived from (22) or from the identity \( (\Sigma - q_1 p_1^*)^{-1} = \Sigma^{-1} q_1 p_1^* \Sigma^{-1}/z \) for \( z \neq 0 \).

All that remains is to apply the following lemma [12] to \( [I, S^{-1} T] \).

**Lemma.**

\[
[I, fg]^p = \begin{bmatrix}
I - \frac{f g}{z} & g(1 + f g^2) \frac{g}{z} \\
g f^2(1 + f g^2) \frac{g}{z}
\end{bmatrix}.
\]

Taking \( f = -\Sigma^{-1} q_1/z \) and \( g = p_2 \), we get \( f^* f = k^* k/z^* z \), and \( g^* g = v^* v \), and hence

\[
L_1^\dagger = \left[ I - \gamma \Sigma^{-1} q_1 q_1^* \Sigma^{-1} \right] S^{-1},
\]

where \( \gamma = v^* v (z^* z + k^* k v v)^{-1} \) and \( \delta = z (z^* z + k^* k v v)^{-1} \). Consequently, from (22) we have

\[
(24) \quad \xi^\dagger = \left[ I - \gamma U^{-1} \Sigma V_1^* \Sigma V_1^* \Sigma^{-1} \right] - \delta U_2 U_2^* B v^* V_1 \Sigma^{-1} U_1^* \cdot [U_1 S^{-1} V_1^*]
\]

which reduces to \( A^\dagger + k h^*/z \) when \( p_2 \) and \( v \) vanish (see [12, ex. 22]).

Case 2. \( q_2 = 0, c \in R(A), z = 0 \). This case is quite different from the previous one since \( S \) is now singular, \( p_1 \neq 0, q_2 \neq 0 \) and

\[
\rho(L_1) = \rho([\Sigma - q_1 p_1^*, q_1 p_2^*]) = \begin{cases}
 r & \text{if } p_2 \neq 0, \\
 r - 1 & \text{if } p_2 = 0.
\end{cases}
\]

Thus we obtain two essentially different cases depending on \( p_2 \). We start by noting that

\[
(25) \quad S \Sigma^{-1} S = S, \quad S^\dagger = (S^\dagger S) \Sigma^{-1} (S S^\dagger),
\]

so that the calculation of \( S^\dagger \) is equivalent to calculating the orthogonal projections \( S^\dagger S \) and \( SS^\dagger \), which is always easier [5]. Indeed, since

\[
(26) \quad R(S) = \ker (\Sigma^{-1} p_1^*), \quad R(S^\dagger) = \ker (\Sigma^{-1} q_1^*)
\]
it follows immediately that
\[ SS^\dagger = I - \Sigma^{-1}q_1(\Sigma^{-1}p_1)^\dagger, \quad S^\dagger S = I - \Sigma^{-1}q_1(\Sigma^{-1}q_1)^\dagger \]
and
\[ S^\dagger = [I - \Sigma^{-1}q_1(\Sigma^{-1}q_1)^\dagger][I - \Sigma^{-1}p_1(\Sigma^{-1}p_1)^\dagger]. \]

Whence for \( p_2 = 0, \)
\[ \zeta^\dagger = U_1S^\dagger V^*_1 = U_1\left[ I - \frac{1}{\Sigma} V_1^* c c^* V_1 \right]^{-1} \left[ I - \frac{1}{\Sigma} U_1^* b b^* U_1 \right] V_1^* = [I - A^\dagger c(A^\dagger c)^\dagger]A^\dagger [I - A^\dagger b(A^\dagger b)^\dagger] = (I - kk^\dagger)A^\dagger (I - hh^\dagger). \]

When \( p_2 \neq 0, \) we may compute \( L^\dagger \) without using the above results, because
\[ L^\dagger = \Sigma[I - \Sigma^{-1}q_1p_1^*, -\Sigma^{-1}q_1p_2^*] \]
which may be g-inverted, since both matrices are of full rank, to give
\[ L^\dagger = [I - \Sigma^{-1}q_1p_1^*, -\Sigma^{-1}q_1p_2^*]^{-1}. \]

We note the following lemma [12, p. 71].

**Lemma.** If \( f^* e = 1, \) then
\[ [-fg^*, 1 - ef^*] = \begin{bmatrix} -g^* f \\ 1 - ee^* \end{bmatrix}. \]

Applying the lemma to \( e = \Sigma^{-1}q_1, f = p_1 \) and \( g = p_2 \) gives (after reversing the order)
\[ L^\dagger_1 = \left[ I - \Sigma^{-1}q_1(\Sigma^{-1}q_1)^\dagger \right]^{-1}. \]

Hence from (22) we obtain
\[ \zeta^\dagger = U_1\Sigma^{-1}V_1^* - U_1\Sigma^{-1}V_1^* c c^* V_1 \Sigma^{-1} U_1^* U_1 \Sigma^{-1} V_1^* - U_2 U_2^* b b^* U_2 \Sigma^{-1} V_1^* = A^\dagger - A^\dagger c(A^\dagger c)^\dagger A^\dagger - (I - A^\dagger A)b b^* (I - A^\dagger A)b = A^\dagger - kk^\dagger A^\dagger - vv^* h^\dagger, \]
which checks with [7].

We may in fact combine both cases when we apply Cline’s results [12] directly to \([S, T]\), and use (27). This yields
\[ (31) \]
\[ L^\dagger = [S, T]^\dagger = \begin{bmatrix} S^\dagger (I - TC^\dagger) \\ C^* \end{bmatrix} = \begin{bmatrix} S^\dagger (I - q_1p_1^\dagger p_2^\dagger p_2) \\ -p_2^\dagger p_1^\dagger \Sigma^{-1} \end{bmatrix} \]
and hence
\[ (32) \]
\[ \zeta^\dagger = (I - kk^\dagger)A^\dagger (I - hh^\dagger) - v v^* h^\dagger, \]
which contains (29) and (30).

On interchanging \( b \) with \( c, h \) with \( k \) and \( u \) with \( v \) and starring we get Case 3.

**Case 3.** \( b \in R(A^\dagger), z \neq 0. \)

\[ (33) \]
\[ \zeta^\dagger = [A^\dagger + kh^\dagger / z][I - \rho hh^\dagger - \sigma uu^\dagger]. \]
where
\[ \rho = u^*u(z^*z + h^*hu^*)^{-1}, \quad \sigma = z(z^*z + h^*hu^*)^{-1}. \]

Case 4. \( b \in R(A^\top), z = 0 \). It follows directly that
\[ \zeta^\top = (I - kb^*c^*u^*)^\top(I - kk^\top)A^\top(I - hh^\top) - hv^\top. \]

We now come to the final case, in which \( p_2 \neq 0, q_2 \neq 0 \). Here we use the factorization
\[ L = \begin{bmatrix} I & q_1q_2^*/\beta \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -q_2p_2^*/\alpha \end{bmatrix} \begin{bmatrix} I \\ p_2p_1^*/\alpha \end{bmatrix}, \]
where \( \alpha = p_2^*p_2, \beta = q_2^*q_2, \) so that \( \rho(M) = \rho(L) = \rho(\Sigma) + 1 = r + 1. \) We may \( g \)-invert this triple product [5], since the products of the first two and last two matrices may each be \( g \)-inverted. Whence
\[ L^\top = \begin{bmatrix} I \\ -p_2p_1^*/\alpha \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & -p_2q_2^*/\alpha \beta \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} \Sigma^{-1} \\ -p_2p_1^*/\Sigma^{-1} \end{bmatrix} \begin{bmatrix} -q_1q_2^*/\beta \\ q_2^*z/\alpha \beta \end{bmatrix}. \]

from which we finally obtain
\[ \zeta^\top = U_1\Sigma^{-1}V_1^* - U_1\Sigma^{-1}q_1q_2^*V_2^*/\beta - U_2p_2p_1^*\Sigma^{-1}V_1^*/\alpha + U_2p_2q_2^*V_2^*z/\alpha \beta \]
\[ = A^\top - A^\top cc^*(I - AA^\top)/\beta - (I - A^\top A)bb^*A^\top \]
\[ + (I - A^\top A)bc^*(I - AA^\top)/\alpha \beta \]
\[ = A^\top - ku^\top - v^\top h^* + zv^\top u^\top. \]

We close with the remark that the techniques developed in this paper may also be used to compute the perturbed Drazin inverses of bordered and rank-one-modified matrices.

REFERENCES


