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Engineering Analysis with Boundary Elements 30 (2006) 790–798

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

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Note on the removal of rigid body motions in the solution of elastostatic traction boundary value problems by SGBEM

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Received 8 February 2005; accepted 3 April 2006
Available online 6 June 2006

Abstract

A study of errors appearing in traction boundary value problems on simply connected domains solved by the symmetric Galerkin boundary element method (SGBEM) is presented. Two methods for the removal of rigid body motions from the nullspace of the discretised SGBEM system matrix, one based on the direct enforcement of additional point supports and the other based on the Fredholm theory of linear operators, are analysed. The fulfillment of the global equilibrium conditions by the discretised load has been found to be the key point in the different behaviour of the errors in displacements obtained applying these two methods. The main objective of this paper is to compare the application of these methods with the SGBEM and with the classical collocation BEM, clarifying in particular a different role of the equilibrium of the discretised load in the SGBEM and classical collocational BEM linear systems. Conclusions of the theoretical analysis presented are confirmed by numerical examples, where the conditions of the global equilibrium are either fulfilled or slightly violated by the discretised load.

Keywords: Symmetric Galerkin boundary element method (SGBEM); Boundary integral equations (BIE); Rigid body motion; Linear elasticity; Fredholm theory

1. Introduction

The application of the symmetric Galerkin boundary element method (SGBEM), e.g., Bonnet et al. [1], in the solution of the traction boundary value problems (TBVP) on the boundary of a simply connected domain is considered. The displacement solution of a TBVP is not uniquely determined because superposing a rigid body motion (RBM), which has zero strain energy and thus also zero stresses, on such a solution, another admissible solution of the problem is obtained. A consequence of this fact is that the square matrix of the discretised SGBEM linear system is theoretically singular (with a non-zero nullspace corresponding to the set of RBMs). Note that this matrix is actually ill-conditioned due to the finite precision arithmetic used by computers. Therefore, reliable methods for removal of the RBMs from the nullspace of the SGBEM system matrix, or in other words methods which will modify the original ill-conditioned SGBEM system to obtain a well-conditioned system, are required.

Several techniques have been developed in the finite and boundary element methods (FEM and BEM) for removal of RBMs from the nullspace of the discretised systems associated to TBVPs (i.e., to achieve invertibility of the modified system) in the past.

Taking into account that the SGBEM [1] has some aspects coincident with those of the FEM, e.g., a variational formulation and a symmetric discretised linear system, it appears to be reasonable to apply in the SGBEM the method successfully used in the FEM, e.g., Szabó and Babuška [2] and Zienkiewicz and Taylor [3]. *Additional point supports* are directly enforced in the displacement field in this method, hereinafter referred to as *Method S*

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doi:10.1016/j.enganabound.2006.04.002

(following the notation introduced by Blázquez et al. [4], *S* being used to address the direct imposition of support conditions). This can be carried out, e.g., by zeroing the appropriate rows and columns in the linear system and defining the corresponding diagonal elements equal to a non-zero number.

The mathematical framework where the above difficulty of TBVP can be well understood is the Fredholm theory of linear operators with zero index [5,6]. Such operators have the dimension of the nullspace and the codimension (dimension of the orthogonal complement to the operator range) finite and equal to each other. In this sense they are similar to square matrices which have the same property. Several methods to achieve invertibility of the modified operator based on the Fredholm theory, hereinafter referred to as *Methods F*, have been developed by various authors in different applications of boundary integral equations (BIEs).

The variant referred to as *Method F1* (following the notation introduced in [4]), in literature sometimes called the *augmenting method* or *bordering method*, has been considered, for example, by Hsiao and Wendland [7], Costabel [8], Karrila and Kim [9], Chen and Zhou [5] and Blázquez et al. [4]. In this method, the original operator is augmented by two operators of finite rank (equal to the dimension of the nullspace of the original operator), one closely related to the vectors of the nullspace and the other to the vectors orthogonal to the range of the original operator, converting the augmented operator into an invertible operator. For an abstract mathematical analysis of this method see Chen and Sun [10] and Chen and Zhou [5].

The variant referred to as *Method F2* (again following the notation introduced in [4]), sometimes called the *completion method*, has been considered, for example, by Heise [11], Ugodchikov and Khutorianskii [12], Power and Miranda [13], Vable [14], Phan-Thien and Tullock [15], Blázquez et al. [4] and Lutz et al. [16]. In this method, an operator of finite rank, obtained as a composition of the finite rank operators considered in *Method F1*, is summed to the original operator in order to obtain an invertible operator.

Relations between *Methods F1* and *F2* in the framework of the classical collocational BEM have been discussed by Kim and Karrila [17] for Stokes flow and by Blázquez et al. [4] for elasticity.

In removing RBMs from the nullspace of the original operator a crucial question arises: *how the solvability or non-solvability of the original discretised system can affect the precision of the solution obtained using the modified invertible system obtained by some of the above mentioned methods*.

This question is related to the following two facts:

- When the solvability condition of the TBVP on the continuum level, i.e., fulfillment of the global equilibrium by the load, is discretised, it may not coincide with the solvability condition of the discretised linear system.
- Although the load prescribed is always equilibrated on the continuum level, after discretisation its global equilibrium can be slightly perturbed.

Due to the above reasons the original discretised linear system may not have a solution for the (equilibrated or not) discretised load. Nevertheless, it is convenient to search for a reasonable approximation of the TBVP solution on the continuum level.

These difficulties, mentioned by Chen and Zhou [5] and Chen and Sun [10], were studied theoretically and numerically by Telles and Paula [18] and Blázquez et al. [4] for the classical collocational BEM. It was shown in [4] that in general there is no equivalence between equilibrium of the discretised load and solvability of the original discretised linear system.

According to the above discussion, *Methods S* and *F* are general methods for the removal of RBMs a priori admitting application not only to the classical collocational BEM (as has been done, e.g., in [4]), but also, providing that the symmetry of the linear system is kept, to the SGBEM. In the present work, a theoretical and numerical analysis of the application of both methods for the removal of the RBMs in SGBEM is introduced. For the sake of brevity the Ref. [4] is heavily relied upon. A theoretical analysis of the *Methods S* and *F* applied to the SGBEM is given in Section 2. It is shown that an equivalence between the global equilibrium of the discretised load and the solvability of the original discretised linear system for a TBVP does exist in the SGBEM, demonstrating in this way a relevant difference between the classical collocation BEM and SGBEM with reference to removal of RBMs. The conclusions of this theoretical analysis are confirmed by numerical examples given in Section 3, where the accuracy of the numerical solutions obtained by *Methods S* and *F* is studied.

2. Removal of RBMs in the SGBEM

2.1. BIEs

Let us consider a bounded elastic body defined by a simply connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a Lipschitz piecewise smooth boundary $\partial\Omega = \Gamma$ (domain Ω being locally on one side of Γ). Let $\Gamma_S \subseteq \Gamma$ denote the smooth part of Γ, i.e., excluding corners, edges, points of curvature jumps, etc. Let the traction operator $\mathscr{T}$ applied to the displacement field $\boldsymbol{u}(x)$ give the traction vector $\boldsymbol{t}(x)$ associated to a unit normal vector $\boldsymbol{n}$ as follows: $\boldsymbol{t}(x) = \mathscr{T}(\boldsymbol{n}, \partial_x)\boldsymbol{u}(x)$.

Let us consider a TBVP with tractions $\boldsymbol{t}_0$ prescribed along the whole boundary Γ. For the sake of simplicity, volume forces are neglected, a generalisation to a TBVP with volume forces being straightforward.

The classical collocational BEM uses the displacement Somigliana identity written for boundary points (denoted here as u-BIE), see [5,19–21]:

$$C_{kl}(x)u_l(x) + \int_\Gamma T_{kl}(x,y)u_l(y)\,\mathrm{d}\Gamma(y) = \int_\Gamma U_{kl}(x,y)t_l(y)\,\mathrm{d}\Gamma(y), \quad x \in \Gamma, \ k,l = 1,\ldots,d, \tag{1}$$

where C_{kl} is the well-known coefficient tensor of the free term [22] ($C_{kl}(x) = \frac{1}{2}\delta_{kl}$ for $x \in \Gamma_S$, δ_{kl} being the Kronecker delta), and the integral kernels U_{kl} and T_{kl}, respectively, represent the fundamental solution of Navier equation in displacements and the corresponding fundamental tractions. The matrix of the fundamental tractions is obtained from the fundamental solution via the traction operator as follows: $\boldsymbol{T}(x,y) = (\mathscr{T}(\boldsymbol{n}(y), \partial_y)\boldsymbol{U}(x,y))^{\mathrm{T}}$, where $\boldsymbol{n}(y)$ denotes the outward normal unit vector to Γ and T stands for the transposed matrix.

An application of the traction operator $\mathscr{T}$ to the displacement Somigliana identity written for off-boundary points, and a subsequent asymptotic procedure, yields the following form of the traction Somigliana identity [5,20,23] for boundary points at Γ_S (denoted here as t-BIE):

$$\oint_\Gamma S_{kl}(x,y)u_l(y)\,\mathrm{d}\Gamma(y) = -\frac{1}{2}t_k(x) + \int_\Gamma T^*_{kl}(x,y)t_l(y)\,\mathrm{d}\Gamma(y), \quad x \in \Gamma_S, \ k,l = 1,\ldots,d, \tag{2}$$

where $\boldsymbol{T}^*(x,y) = \mathscr{T}(\boldsymbol{n}(x), \partial_x)\boldsymbol{U}(x,y)$ and $\boldsymbol{S}(x,y) = \mathscr{T}(\boldsymbol{n}(x), \partial_x)\boldsymbol{T}(x,y)$.

In (1) and (2) the integrals with the strongly singular integral kernels T_{kl} and T^*_{kl} are evaluated in the sense of Cauchy principal value, and the integral with the hypersingular integral kernel S_{kl} is evaluated in the sense of Hadamard finite part, both considering a spherical vanishing zone, see for details [5,23,24].

With reference to the above-defined integral kernels, let us recall the following reciprocity relations [1]:

$$\boldsymbol{U}(x,y) = \boldsymbol{U}^{\mathrm{T}}(y,x), \quad \boldsymbol{T}^*(x,y) = \boldsymbol{T}^{\mathrm{T}}(y,x), \quad \boldsymbol{S}(x,y) = \boldsymbol{S}^{\mathrm{T}}(y,x). \tag{3}$$

Additionally, the fundamental solution is symmetric, i.e., $\boldsymbol{U}(x,y) = \boldsymbol{U}^{\mathrm{T}}(x,y)$, and, according to [25], the hypersingular integral kernel $\boldsymbol{S}$ in 2D is also symmetric, i.e., $\boldsymbol{S}(x,y) = \boldsymbol{S}^{\mathrm{T}}(x,y)$.

A general SGBEM formulation involves a combination of both aforementioned BIEs, depending on the boundary conditions prescribed. In the present case of a TBVP, solely t-BIE (2) is required, the symmetric integral operator of the first kind of the SGBEM system being then defined by the hypersingular integral on the left-hand side of (2). Recall that the non-symmetric integral operator of the second kind defined by the integral on the left-hand side of (1) appears in the classical BEM system for a TBVP.

2.2. Removal of RBMs and global equilibrium

An analysis of the errors which may be present in the numerical solution of TBVP, in the case of equilibrated and non-equilibrated discretised loads, by the SGBEM and also the classical BEM is presented.

Let us denote a basis of the linear space of RBM by $\boldsymbol{\mu}^\alpha$ for $x \in \Omega \cup \Gamma$, where $\alpha = 1,\ldots,n_d$ with $n_2 = 3$ (in 2D) and $n_3 = 6$ (in 3D). This basis can be defined, e.g., in 2D, by the following vectors:

$$\boldsymbol{\mu}^1(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\mu}^2(x) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \boldsymbol{\mu}^3(x) = \begin{Bmatrix} -x_2 \\ x_1 \end{Bmatrix}, \tag{4}$$

where $\boldsymbol{\mu}^1$ and $\boldsymbol{\mu}^2$ are rigid body translations and $\boldsymbol{\mu}^3$ is a rigid body rotation. Note that there are three linearly independent rigid body translations and three linearly independent rigid body rotations in 3D.

Applying a RBM as a solution of the auxiliary problem in the second Betti theorem of reciprocity of work [5,21] written for Ω results in the following global equilibrium conditions for any traction solution of a BVP on Ω:

$$\int_\Gamma t_k(x)\mu^\alpha_k(x)\,\mathrm{d}\Gamma(x) = 0, \quad \alpha = 1,\ldots,n_d. \tag{5}$$

A crucial property of RBMs for the present work follows from their substitution, together with the associated zero tractions, into (1) and (2), giving

$$\frac{1}{2}\mu^\alpha_k(x) + \int_\Gamma T_{kl}(x,y)\mu^\alpha_l(y)\,\mathrm{d}\Gamma(y) = 0, \quad x \in \Gamma_S, \tag{6a}$$

$$\oint_\Gamma S_{kl}(x,y)\mu^\alpha_l(y)\,\mathrm{d}\Gamma(y) = 0, \quad x \in \Gamma_S. \tag{6b}$$

According to (6) RBMs are in the nullspace of the integral operators of both approaches, the classical BEM and SGBEM, for TBVPs, and thus the BEM solution is non-unique. An analysis of techniques which remove this non-uniqueness from the SGBEM solution, keeping the symmetry of the final linear system, is the aim of the present work.

In addition to the simple *Method S* which mimics the usual FEM approach [4] (note that this method is called *Method S1* in [4]), directly imposing additional point supports in the SGBEM linear system to remove the RBMs, two variants of *Methods F*, which rely on the Fredholm theory, will be studied.

Method F1 modifies (2) by 'augmenting' the original BIE system:

$$\oint_\Gamma S_{kl}(x,y)u_l(y)\,\mathrm{d}\Gamma(y) + \sum_{\alpha=1}^{n_d}\mu^\alpha_k(x)\omega^\alpha = -\frac{1}{2}t_k(x) + \int_\Gamma T^*_{kl}(x,y)t_l(y)\,\mathrm{d}\Gamma(y), \tag{7}$$

$$\int_\Gamma \mu^\alpha_l(y)u_l(y)\,\mathrm{d}\Gamma(y) = 0, \quad \alpha = 1,\ldots,n_d, \tag{8}$$

displacements $\boldsymbol{u}$ and parameters ω^α being the unknowns of the augmented linear system.

Method F2 modifies (2) by 'completing' the original BIE system:

$$\oint_\Gamma S_{kl}(x,y)u_l(y)\,\mathrm{d}\Gamma(y) + \sum_{\alpha=1}^{n_d}\mu_k^\alpha(x)\int_\Gamma \mu_l^\alpha(y)u_l(y)\,\mathrm{d}\Gamma(y) = -\frac{1}{2}t_k(x) + \int_\Gamma T_{kl}^*(x,y)t_l(y)\,\mathrm{d}\Gamma(y), \tag{9}$$

displacements $\boldsymbol{u}$ being the only unknowns.

It can be shown, see for an analogous analysis [4,5,10,17], that the modified systems (7)–(9) always have a unique solution, and that the numerical solutions obtained by *Methods F1* and *F2* coincide with the exception of round-off errors.

In what follows the question of solvability of the original equations (1) and (2) and their discretised versions will be discussed.

Let us denote $\boldsymbol{\tau}^\alpha$ the solution of the exterior displacement BVP, defined on the complementary domain to Ω, with boundary conditions $\boldsymbol{u} = \boldsymbol{\mu}^\alpha$ and with the far-field behaviour given by the relation:

$$u_k(x) = U_{kl}(x,0)b_l + \mathrm{O}(\|x\|^{1-d}), \quad \|x\| \to \infty, \quad \text{where } b_k = \int_\Gamma t_k(y)\,\mathrm{d}\Gamma(y). \tag{10}$$

BIE (1) rewritten for this BVP takes the form [4]:

$$\mu_k^\alpha(x) = \int_\Gamma U_{kl}(x,y)\tau_l^\alpha(y)\,\mathrm{d}\Gamma(y), \quad x \in \Gamma, \ k,l = 1,\ldots,d. \tag{11}$$

Solution of this BIE always exists and is unique in 3D, whereas it exists and is unique except for (either one or two) critical scales of the domain considered in 2D, see [26] for details.

As was shown in [4], the left-hand side of (1) is always orthogonal to $\boldsymbol{\tau}^\alpha$ (considering a natural generalisation of the inner product in the space of square Lebesgue-integrable functions $[L_2(\Gamma)]^d$). However, if the prescribed tractions $\boldsymbol{t}_0$ did not exactly satisfy the equilibrium condition (5), the right-hand side of (1) would not be orthogonal to $\boldsymbol{\tau}^\alpha$. In such a case the original equation (1) does not have any solution, whereas its modified versions, with an invertible operator on the left-hand side obtained by the *Methods F* [4], do have.

This situation is somewhat different when a discretised version of u-BIE (1) is considered. Then, the condition of equilibrium of the discretised load is in general not equivalent to the condition of solvability of the corresponding linear system. As was explained in [4], this is related to the fact that $\boldsymbol{\tau}^\alpha$ in general cannot be exactly approximated by the boundary element functions and also to the usually collocational nature of the classical BEM linear system. The difference between the numerical solution of the modified u-BIE and the solution of the original TBVP most noticeably appears in *Method S*, see [4], for the reason that in this method some collocation equations are in fact dropped out of this linear system, consequently not all of them being satisfactorily fulfilled.

If, on the other hand, SGBEM system, given by (2), is used for the same TBVP, the left-hand side of the equation is orthogonal to $\boldsymbol{\mu}^\alpha$, according to (6b) and the reciprocity relation $(3)_3$, i.e.,

$$\int_\Gamma \mu_k^\alpha(x)\left(\oint_\Gamma S_{kl}(x,y)u_l(y)\,\mathrm{d}\Gamma(y)\right)\mathrm{d}\Gamma(x) = \int_\Gamma u_l(y)\left(\oint_\Gamma S_{lk}(y,x)\mu_k^\alpha(x)\,\mathrm{d}\Gamma(x)\right)\mathrm{d}\Gamma(y) = 0, \tag{12}$$

while for the right-hand side, when multiplied by $\boldsymbol{\mu}^\alpha$ (from the left) it holds, using (6a) and the reciprocity relation $(3)_2$, that

$$\begin{aligned}\int_\Gamma \mu_k^\alpha(x)\left(-\frac{1}{2}t_k(x) + \int_\Gamma T_{kl}^*(x,y)t_l(y)\,\mathrm{d}\Gamma(y)\right)\mathrm{d}\Gamma(x) &= \int_\Gamma t_l(y)\cdot\left(-\frac{1}{2}\mu_l^\alpha(y) + \int_\Gamma T_{lk}(y,x)\mu_k^\alpha(x)\,\mathrm{d}\Gamma(x)\right)\mathrm{d}\Gamma(y) \\ &= -\int_\Gamma t_l(y)\mu_l^\alpha(y)\,\mathrm{d}\Gamma(y).\end{aligned} \tag{13}$$

This means that the right-hand side of (2) is also orthogonal to $\boldsymbol{\mu}^\alpha$ for equilibrated loads. Thus, the same kind of errors as appeared in the solution of u-BIE may also occur when solving an SGBEM system if the load is not equilibrated.

Notice that the order of integration has been changed in Eqs. (12) and (13), which include principal value and finite part integrals, such a procedure also being allowed in this case as shown, e.g., by Bonnet [27].

Inasmuch as $\boldsymbol{\mu}^\alpha$ can be approximated exactly using boundary element shape functions (when considering linear or higher order isoparametric boundary elements), the discretised version of (2) has a solution for any equilibrated discretised load and does not have a solution for any non-equilibrated load.

Therefore, the kind of error discussed above may appear in the solution of the discretised SGBEM system, but only for non-equilibrated discretised external loads, e.g., due to a complicated behaviour of prescribed tractions along the boundary.

3. Numerical examples

The results of *Method S* and *Method F2* for removal of the non-uniqueness in the solution of the SGBEM will be compared. Recall that the results of *Method F1* would be the same as those presented for *Method F2*, with the exception of the rounding-off errors.

In particular, the appearance and behaviour of the errors discussed in the previous section will be analysed. Two typical examples have been chosen to study the behaviour of the SGBEM solutions of TBVPs on bounded domains. First a problem whose traction distribution can be represented exactly by the boundary element shape functions used, and then a problem where this condition is not satisfied, due to a higher order of variations of tractions along the boundary, will be solved. The former case is represented by a plate bending problem, the latter by a piece of a ground subjected to a half-line load. Note that the same problems were studied in Blázquez et al. [4] using the collocation BEM solving u-BIE (1). This will allow an easy and direct comparison of the results obtained using both BEM approaches.

Plane strain state is assumed in both cases. Material parameters Young's modulus and Poisson's ratio are $E = 100\,\mathrm{GPa}$ and $\nu = 0.25$, respectively. Dimensions of solids and displacements are given in millimeters. Load parameter p is set to 100 MPa.

The problems presented have been solved by a 2D SGBEM code. The code uses straight linear continuous elements with nodes at the ends of the element allowing discontinuity of tractions when required [21]. Analytical integrations are used to evaluate the influence matrices. Gauss elimination method is applied to solve the SGBEM linear equation system. In all calculations double precision floating point arithmetic (REAL(8) in FORTRAN90) is used.

3.1. Example 1: equilibrated discretised load

The accuracy of the SGBEM solutions in the case of the equilibrated discretised loads is evaluated in this example. As follows from an analysis due to Blázquez et al. [4], significant errors appear applying *Method S* to the classical u-BIE, even in the present example in which the discretised external load is in equilibrium. Nevertheless, as will be shown here, this is not the case of *Method S* applied to the SGBEM in the case of equilibrated discretised external loads.

Let us consider the solution of a plate subjected to bending [4], see Fig. 1, whose Airy stress function is expressed as [28]: $F(x_1, x_2) = (p/3h)x_2^3$, where $h = 40\,\mathrm{mm}$ is the plate height.

The additional point supports are placed at points $P_1(-40, 0)$ and $P_2(40, 0)$ in the following manner:

$$u_1(P_1) = 0, \quad u_2(P_1) = 0, \quad u_2(P_2) = 0. \tag{14}$$

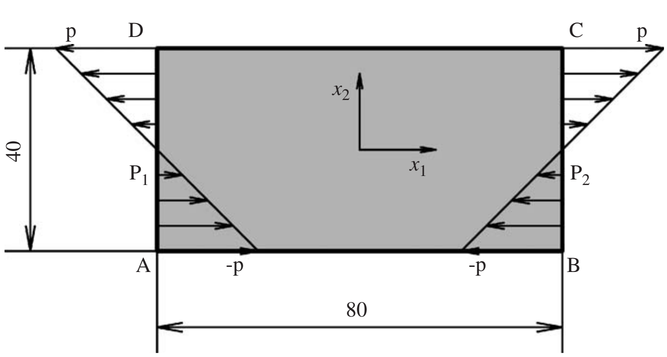

Fig. 1. Geometry of the plate bending problem.

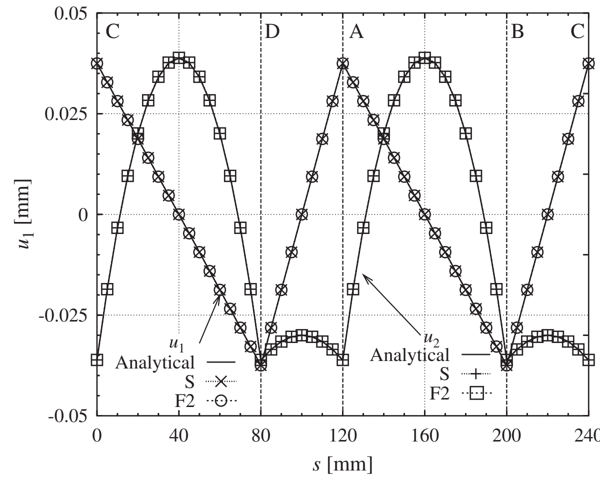

Fig. 2. Displacement results for the plate bending problem.

Recall that in *Method F2*, resultant displacements, which fulfill these point support conditions, are obtained by adding a suitable RBM to the solution in displacements in a post-processing step.

The discretisation used is also the same as in the aforementioned paper [4], a uniform mesh with 16 elements at faces AB and CD and 8 elements at BC and DA being used.

Displacements calculated by the *Methods S* and *F2* are plotted in Fig. 2, s being the arc length measured starting from the point C in counterclockwise direction. The results obtained agree excellently with the analytical values. Thus, as predicted by the theoretical analysis in Section 2, no significant errors appear in the *Method S* applied to the SGBEM in presence of an equilibrated discretised load. The errors normalised by the maximum value of the analytical solution are shown in Fig. 3. An absolute agreement between both *Methods*, *S* and *F2*, might at first sight be considered surprising, but it is just in line with the following two facts. First, RBMs are approximated exactly by the discretisation used. Second, for equilibrated loads the right-hand sides of Eqs. (2) and (9) are orthogonal to all RBMs, and thus the second term on the left-hand side of (9) must be equal to zero. These results would allow application of *Method S* in the SGBEM (at least for problems with equilibrated discretised loads), as opposed to the classical collocation BEM where, as was shown by Blázquez et al. [4], errors for this method are not negligible, peaks appearing in the displacement solution at the additional point supports.

3.2. Example 2: non-equilibrated discretised load

This example differs from the previous one in the fact that the total load applied loses equilibrium when discretised. According to the analysis in Section 2 larger errors, in comparison with the previous example, can be expected in *Method S* applied to the SGBEM for a load

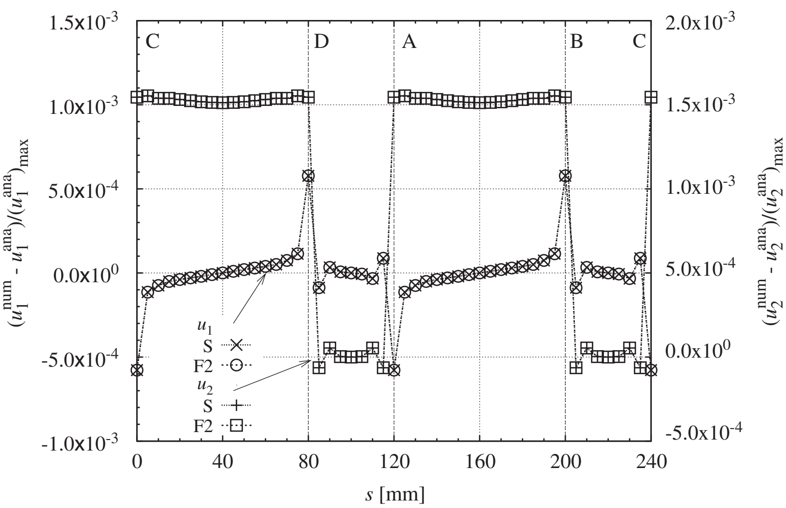

Fig. 3. Normalised errors for the plate bending problem.

which cannot be exactly represented by the boundary element functions.

Let us consider a domain surrounded by a quadrangle $ABCD$ [4,21], see, Fig. 4, in the half-plane ($x_2 > 0$) subjected along the negative half-axis x_1 to a constant pressure p. The Airy stress function for this problem is expressed as [28]:[1]

$$F(x_1, x_2) = -\frac{p}{2\pi}\left[-x_1x_2 + (x_1^2 + x_2^2)\mathrm{Arctan}\left(\frac{x_2}{x_1}\right)\right],$$
$$\text{where} \quad \mathrm{Arctan}x = \begin{cases} \arctan x, & \text{if } x \geqslant 0, \\ \pi + \arctan x, & \text{if } x < 0. \end{cases} \tag{15}$$

The point support conditions for *Method S* (imposed also on the solution of *Method F2* in a post-processing step as explained above) are placed as follows:

$$u_1(60,0) = 0, \quad u_2(60,0) = 0, \quad u_2(150,0) = 0. \tag{16}$$

The BEM mesh applied here is also identical to that in Blázquez et al. [4], faces AB, BC, CD and DA, respectively, being discretised by 3, 5, 7 and 7 elements of equal length along each face.

The displacements calculated by the SGBEM using *Methods S* and *F2* do not coincide in this case. These differences can be clearly seen in Fig. 5, s being the arc length measured starting from the point A in counter-clockwise direction. Agreement between the results by *Method F2* and the analytic solution is excellent, taking into account the coarse discretisation used. It should be remarked that similar results were obtained by *Method F1* applied to the classical collocational BEM in [4]. Peaks appear, however, at the support points (given by $s = 134.4$ and 224.4 mm) in the results of *Method S*, the differences between the results of the *Methods S* and *F2*, along a major

[1]Notice that in Refs. [4,21] this function was erroneously considered to be multiplied by π.

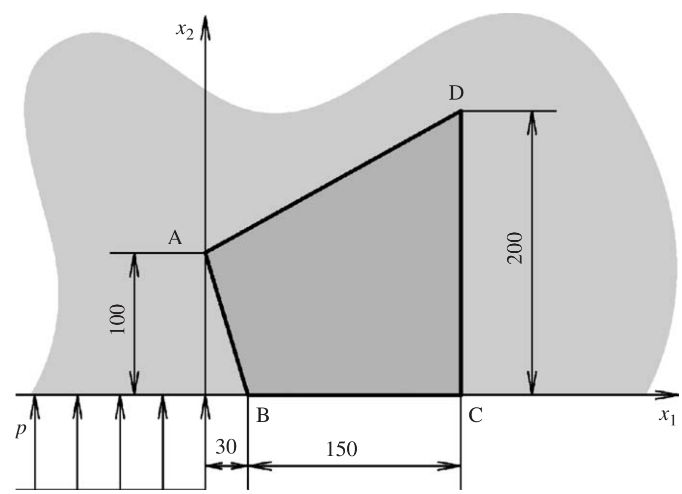

Fig. 4. Configuration of the problem based on the half-plane solution.

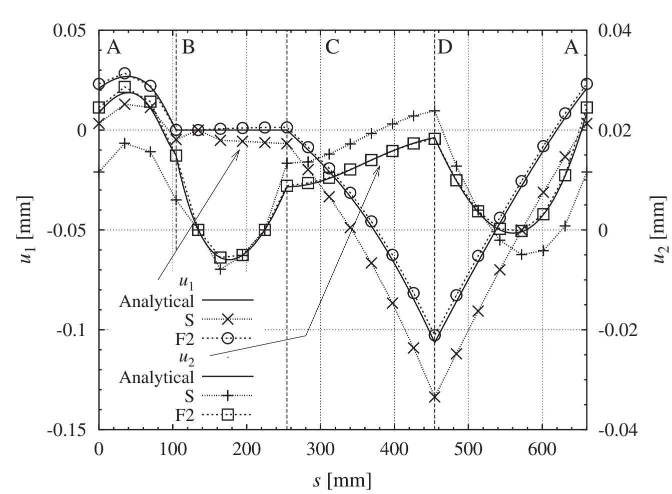

Fig. 5. Displacement results for the half-plane-solution problem, coarse mesh.

part of the boundary, being approximately given by a relative RBM (translation and rotation).

These effects (peaks and a relative RBM) can be even more strikingly seen for a finer mesh, though the absolute differences between both *Methods*, *S* and *F2*, are smaller due to a smaller discretisation error. The boundary is discretised by 21, 30, 40 and 41 elements at faces *AB*, *BC*, *CD* and *DA*, respectively, in order to make the mesh along the whole boundary almost uniform. The distribution of displacements is now in excellent agreement with the analytical solution, see Fig. 6. However, the distinction between the results of the *Methods S* and *F2*, see the normalised errors in Fig. 7, keeps the same character as for the coarse mesh, where, for the sake of brevity, an analogous figure has been omitted because the errors are clearly evident from Fig. 5. On one hand, errors of *Method F2* are very small and smooth at each face, and on the other hand, errors of *Method S* are relatively large along the boundary and non-smooth at the support points. The nearly linear distribution of the errors in *Method S* at the faces with no point support indicates an approximate RBM appearing in the results of this method again, its magnitude being, however, substantially smaller than in the case of the coarse mesh. The behaviour of the results by this method at the supported face does not follow such a simple distribution inasmuch as these errors vanish exactly at the support points, which produces peaks in the error distribution at these points, the solution obtained being consequently locally perturbed along this face. Note that this conclusion coincides with that found for *Method S* applied in the classical BEM approach [4]: the load, which cannot be accurately represented by the boundary elements and consequently in its discretised form generally violates the global equilibrium, results in strong peaks of errors in displacements in the vicinity of the additional support points, although the actual errors vanish at these points. It is clear that if stresses are evaluated using such displacements, either on the boundary (the so-called in-boundary stresses) or at interior points, relatively large errors can be expected near the support points due to these peaks.

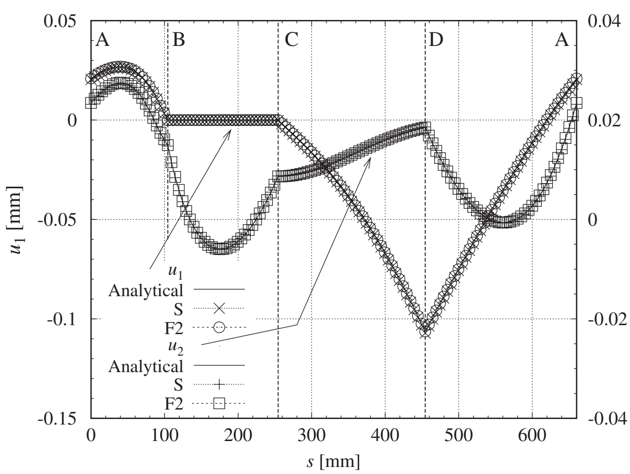

Fig. 6. Displacement results for the half-plane-solution problem, fine mesh.

The above-observed relatively poor behaviour of *Method S* in contrast with *Method F2*, in the case where the discretised load is non-equilibrated and thus the original discretised SGBEM system does not have any solution, can be attributed to the following fact: in *Method S* the errors in the fulfillment of the linear equations of the original SGBEM system are concentrated in a few products of the BIE (2) with the element shape functions associated to the additional point supports (equations are dropped out from the final system), whereas in *Method F2* these errors are distributed in some way between all the linear equations.

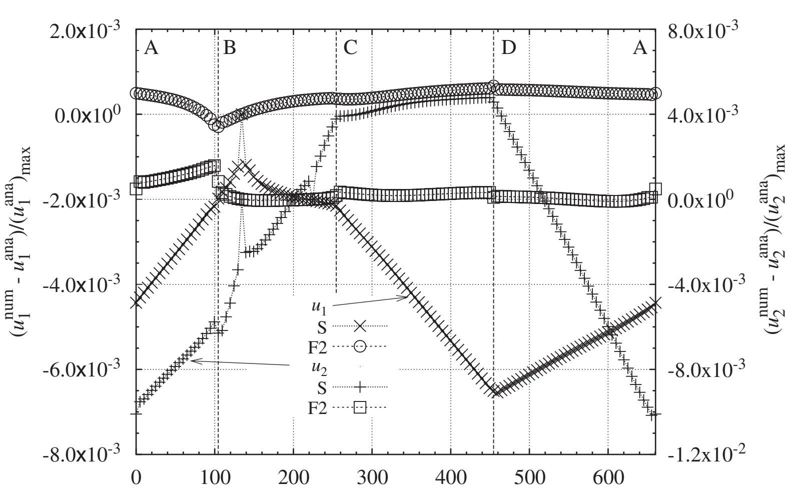

Fig. 7. Normalised errors for the half-plane solution problem, fine mesh.

4. Conclusions

The errors in solving TBVPs on bounded domains by the SGBEM have been studied. An equivalence between equilibrium of the discretised load and solvability of the original discretised SGBEM linear systems has been explained and demonstrated. It has to be stressed that such an equivalence does not exist in the classical collocational BEM. The theoretically predicted differences in the error behaviour between configurations with equilibrated and non-equilibrated discretised loads are observed in the numerical examples.

Two kinds of methods, referred to as *Method S* (which directly imposes the point support conditions in the linear system) and *Methods F* (based on the Fredholm theory of linear operators), for removal of the RBMs from the nullspace of matrix of the SGBEM system, which keep the final system symmetric, have been proposed, analysed theoretically and tested numerically.

Although *Method S*, widely used in the FEM, gives excellent results for problems with equilibrated discretised loads (as, e.g., in the presented bending problem where the load varies linearly), it is not reliable in solving problems with a complex behaviour of traction distribution, where a slight violation of the equilibrium of the discretised load may appear (as, e.g., in the presented half-plane-solution problem). In such cases, it provides an acceptable solution for very fine meshes only. It has to be mentioned with reference to the application of *Method S* to problems with equilibrated discretised loads, that whereas it provides satisfactory results in the SGBEM, peaks appear in the results obtained applying the classical collocational BEM [4].

According to the theoretical analysis presented and numerical results obtained, *Methods F*, based on the Fredholm theory, may be recommended as reliable methods for removal of RBMs in the SGBEM, in the same manner as has been done in [4] for the classical collocational BEM, as they produce accurate results in both cases, with equilibrated and non-equilibrated discretised loads.

Note finally that more sophisticated approaches for solution of non-uniqueness of TBVPs in multiple connected domains (with one or several cavities) are proposed and studied in [29]. These approaches also deal with other BVPs different from TBVPs on multiple connected domains, which also have a non-unique solution when solved by SGBEM.

Acknowledgement

Part of the research presented was performed when R. V. was on NATO scientific fellowship (OTAF03-054) at the School of Engineering, University of Seville. The authors also acknowledge the financial support from the Scientific Grant Agency of the Slovak Republic (Grant no. 1/1089/04) given to R. V. and from the Spanish Ministry of Science and Technology (Project no. MAT2003-03315) given to V. M. and F. P.

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