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Framework for biorthogonal Fourier series

(differential equations/elasticity/Stokes flow)

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ABSTRACT **Biorthogonal Fourier series occur when applying separation of variables to many problems. Here an approach which possesses considerable advantages with respect to the standard one is explained.**

Biorthogonal Fourier series for fourth-order equations have received much attention in recent years (1, 2). The origin of the method can be traced to Smith (3). Recently, Joseph (2) showed the considerable generality of the method. An independent approach was followed by Herrera to derive orthogonality relationships for Rayleigh waves (4, 5, 6).

In this article, this latter approach is generalized to give a theoretical framework for biorthogonal Fourier series. The setting for this framework is a recently developed algebraic theory of boundary value problems (7, 8). The theory is explained in connection with an introductory example and then it is developed systematically.

AN INTRODUCTORY EXAMPLE

To fix ideas, let us consider a simple example. Let $u(x,y)$ and $v(x,y)$ be solutions of the biharmonic equation in a horizontal strip; i.e.,

$$\Delta^2 u = \Delta^2 v = 0,\ -1 < y < 1,\ -\infty < x < \infty \quad [1a]$$

such that

$$u = v = 0; \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \text{ at } y = \pm 1. \quad [1b]$$

Then, one can define an antisymmetric bilinear functional A_0 by using

$$\langle A_0 u,v\rangle = \int_{-1}^{1} \left\{ v\frac{\partial \Delta u}{\partial x} - \Delta u \frac{\partial v}{\partial x} + \Delta v \frac{\partial u}{\partial x} - u\frac{\partial \Delta v}{\partial x} \right\}_{x=\xi} dy, \quad [2]$$

where $-\infty < \xi < +\infty$. Well-known reciprocity relationships for the biharmonic equation imply that the expression for A_0 given by Eq. **2** is independent of ξ whenever Eq. **1a** and **b** is satisfied.

Separable solutions satisfy (2)

$$\phi_n(x,y) = f_n(y)e^{-\lambda_n x}, \quad [3]$$

where

$$\sin^2 2\lambda_n - 4{\lambda_n}^2 = 0. \quad [4]$$

It can be shown that $R_e\lambda_n \neq 0$ and that $-\lambda_n$ is a root whenever λ_n satisfies Eq. **4**.

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From Eq. **2**, it is seen that

$$\langle A_0\phi_n,\phi_m\rangle = e^{-(\lambda_n+\lambda_m)\xi} \int_{-1}^{1} \left\{ \phi_m \frac{\partial \Delta\phi_n}{\partial x} - \Delta\phi_n \frac{\partial \phi_m}{\partial x} + \Delta\phi_m \frac{\partial \phi_n}{\partial x} - \phi_n \frac{\partial \Delta\phi_m}{\partial x} \right\}_{x=0} dy, \quad [5]$$

which holds for every ξ. Hence, $\lambda_n + \lambda_m \neq 0 \Rightarrow \langle A_0\phi_n,\phi_m\rangle = 0$.

Let us restrict the definition [Eq. **3**] by the condition $\mathrm{Re}\lambda_n \geq 0$ and introduce the notation $(n \geq 1)$

$$\phi_n^*(x,y) = f_n^*(y)e^{\lambda_n x}. \quad [6]$$

Then, it can be shown that

$$\langle A_0\phi_n,\phi_n^*\rangle \neq 0. \quad [7]$$

The notation established by Eqs. **3** and **6** classifies separable solutions into two disjoint groups. Define N_P^1 as the linear manifold of functions spanned by the system $\{\phi_1,\phi_2,\ldots\}$, and define N_P^2 correspondingly, by using $\{\phi_1^*,\phi_2^*,\ldots\}$ instead. Let

$$N_P = N_P^1 + N_P^2. \quad [8]$$

The properties characterizing the subspaces N_P^1 and N_P^2 are

$$u \to 0 \quad \text{as } x \to +\infty \quad \text{whenever} \quad u \in N_P^1 \quad [9a]$$

$$u \to 0 \quad \text{as } x \to -\infty \quad \text{whenever} \quad u \in N_P^2. \quad [9b]$$

The null subspace N_{Ao} of A_0 will be needed. This is

$$N_{Ao} = \{u \in N_P | \langle A_0 u,v\rangle = 0\ \forall\ v \in N_P\}. \quad [10]$$

It can be seen that the only function belonging to this space is the zero function; i.e.,

$$N_{Ao} = \{0\}. \quad [11]$$

We recall the following properties of these spaces.

(*i*) N_P^1 and N_P^2 are commutative subspaces; i.e.,

$$\langle A_0 u,v\rangle = 0\ \forall\ u \in N_P^\alpha \quad \text{and} \quad v \in N_P^\alpha; \quad \alpha = 1,2. \quad [12]$$

(*ii*) $N_P^\alpha \supset N_{Ao};\ \alpha = 1,2.$ **[13]**

(*iii*) Given $u \in N_P$,

$$\langle A_0 u,v\rangle = 0,\ \forall\ v \in N_P^\alpha \Rightarrow u \in N_P^\alpha; \quad \alpha = 1,2. \quad [14]$$

(*iv*) For every $u \in N_P$, there exist elements $u_1 \in N_P^1$ and $u_2 \in N_P^2$ such that

$$u = u_1 + u_2. \quad [15]$$

(*v*) $N_{Ao} = N_P^1 \cap N_P^2.$ **[16]**

Although property **13** is trivially satisfied in this case, it is listed here because later it will be generalized to include cases in which Eq. 11 does not hold and then [13] is no longer a trivial requirement.

Commutative subspaces satisfying [13] will be said to be regular, and completely regular when, in addition, property 14 is fulfilled (7). When two regular subspaces span the space; i.e., when Eq. 8 is fulfilled, one says that the pair $\{N_P^1, N_P^2\}$ constitutes a canonical decomposition of the space N_P. It can be shown (7) that, when $\{N_P^1, N_P^2\}$ is a canonical decomposition, both N_P^1 and N_P^2 are necessarily completely regular subspaces, which satisfy property 16.

We recall that the families $\mathcal{B} = \{\phi_1, \phi_2, \ldots\} \subset N_P^1$ and $\mathcal{B}^* = \{\phi_1^*, \phi_2^*, \ldots\} \subset N_P^2$ have the following property. Given any $u \in N_P$, one has

$$\langle A_0 u, \phi_n \rangle = 0, \; n = 1,2, \ldots \Rightarrow u \in N_P^1 \qquad [17a]$$

and

$$\langle A_0 u, \phi_n^* \rangle = 0, \; n = 1,2, \ldots \Rightarrow u \in N_P^2. \qquad [17b]$$

Families of functions satisfying either [17a] or [17b] are called *c*-complete (9) for N_P^1 and N_P^2, respectively.

From Eqs. 6 and 7, it follows that

$$\langle A_0 \phi_n, \phi_m^* \rangle = 0 \qquad \text{if } n \neq m \qquad [18]$$

and

$$\langle A_0 \phi_n, \phi_n^* \rangle \neq 0 \qquad \text{if } n = 1,2, \ldots. \qquad [19]$$

Hence, multiplying each of the functions of the families of separable solutions by suitable constants, one can assume that

$$\langle A_0 \phi_n, \phi_m^* \rangle = \delta_{nm}. \qquad [20]$$

When [18] is satisfied, one says that the families $\{\phi_1, \phi_2, \ldots\} \subset N_P^1$ and $\{\phi_1^*, \phi_2^*, \ldots\} \subset N_P^2$ are biorthogonal. If, in addition, [20] is fulfilled, the families are said to be biorthonormal.

Now, any function $u \in N_P$ can be written as

$$u = \sum_{n=1}^{\infty} a_n \phi_n + \sum_{n=1}^{\infty} b_n \phi_n^*. \qquad [21]$$

It is convenient to recall that each of the systems of constants a_n, b_n ($n = 1,2, \ldots$) possesses only a finite number of nonvanishing elements, because N_P^1 and N_P^2 have been defined as the linear manifolds spanned by separable solutions. Later we will consider actual infinite series, but this has been avoided here to keep this introductory example sufficiently simple. When the systems $\{\phi_1, \phi_2, \ldots\}$ and $\{\phi_1^*, \phi_2^*, \ldots\}$ are biorthonormal, it is straightforward to verify that

$$a_n = \langle A_0 u, \phi_n^* \rangle \qquad [22a]$$

and

$$b_n = \langle A_0 \phi_n, u \rangle. \qquad [22b]$$

By using the notions thus far introduced, we can summarize our results as follows:

The space of biharmonic functions N_P defined in a horizontal strip admits a canonical decomposition into two subspaces $\{N_P^1, N_P^2\}$ such that the families of separable solutions $\{\phi_1, \phi_2, \ldots\} \subset N_P^1$ and $\{\phi_1^*, \phi_2^*, \ldots\} \subset N_P^2$ are *c*-complete for N_P^1 and N_P^2, respectively. Even more, these two systems are biorthogonal and, by a suitable choice, they can be taken to be biorthonormal. In this case, any function of the space N_P can be represented by Eq. 21, where the coefficients a_n and b_n ($n = 1,2, \ldots$) are given by [22].

This paper is devoted to an explanation of how this simple scheme can be formulated to apply to a general class of partial differential equations relevant in continuum mechanics and other fields. In the simple introductory example given above, the space N_P is not equipped with a topological structure. However, in more general situations, topological considerations will have to be included.

PRELIMINARY RESULTS AND NOTATION

It is often useful to associate bilinear functionals with partial differential equations for their study. Given a linear space D, such bilinear functionals can also be thought as operators $P:D \to D^*$, where D^* is the algebraic dual of D; i.e., D^* is the linear space whose elements $\alpha \in D^*$ are linear functionals $\alpha: D \to \mathcal{F}$. For the purpose of the discussion that follows, $\mathcal{F}$ will be the field of either real or complex numbers.

For example, consider a region R, of the euclidean space $\mathbf{R}^n$ ($n \geq 1$), and let the linear space $D = \mathcal{C}^\infty(R)$, where $\mathcal{C}^\infty(R)$ is the space of infinitely differentiable functions in R. Alternatively, one could take D as the Sobolev space $H^s(R)$ of any order $s \geq 3/2$. Define $P:D \to D^*$ so that, for any $u \in D$ and $v \in D$, one has

$$\langle Pu, v \rangle = \int_R v \nabla^2 u \, dx \qquad [23]$$

Given any bilinear functional $P:D \to D^*$, its transpose will be denoted by $P^*:D \to D^*$. Therefore, $A = P - P^*$ is an antisymmetric bilinear functional. When P is formally symmetric, A will be a boundary operator. For example, when P is given by Eq. 23,

$$\langle Au, v \rangle = \int_{\partial R} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} dx, \qquad [24]$$

which involves only boundary values. We will denote by N_A the null subspace of A (i.e., $N_A = \{u \in D | \langle Au, v \rangle = 0 \; \forall \; v \in D\}$). It is then easy to see that, for the example [24], $N_A = \{u \in D | u = (\partial u / \partial n) = 0 \text{ on } \partial R\}$.

The notions of regular and completely regular subspaces (8), as well as canonical decompositions of D, will be required to understand what follows.

Definition 1: A linear subspace $I \subset D$ is said to be regular for $A:D \to D^*$ (or for P) when

$$(i) \qquad \langle Au, v \rangle = 0 \; \forall \; u \in I \quad \text{and} \quad v \in I \qquad [25]$$

$$(ii) \qquad I \supset N_A. \qquad [26]$$

One says that I is completely regular for A if, in addition, *(iii)* for every $u \in D$,

$$\langle Au, v \rangle = 0 \; \forall \; v \in I \Rightarrow u \in I. \qquad [27]$$

This definition of completely regular subspace is equivalent to the condition that $I \subset D$ enjoys the property:

$$\langle Au, v \rangle = 0 \; \forall \; v \in I \Leftrightarrow u \in I. \qquad [28]$$

Definition 2: Let $\{I_1, I_2\}$ be subspaces of D that are regular for $A:D \to D^*$. Then, the ordered pair $\{I_1, I_2\}$ is said to be a canonical decomposition of D, with respect to A when

$$D = I_1 + I_2 \qquad [29]$$

It has been shown (8, 9), that such canonical decompositions possess the following important property.

THEOREM 1. *Let* $\{I_1, I_2\}$ *be a pair of subspaces of* D. *Then* $\{I_1, I_2\}$ *constitute a canonical decomposition of* D *with respect* *to* A, *if and only if* I_1 *and* I_2 *are completely regular and*

$$D = I_1 + I_2;\ N_A = I_1 \cap I_2. \quad [30]$$

CANONICAL DECOMPOSITIONS OF THE SPACE OF SOLUTIONS

When $P:D \to D^*$ is associated with a differential equation, the homogeneous equation is $Pu = 0$. Thus, the space of solutions of the homogeneous equation is the null subspace N_P of P.

When $A_1:D \to D^*$ and $A_2:D \to D^*$ are antisymmetric operators such that (*i*)

$$A = A_1 + A_2 \quad [31a]$$

and (*ii*) A_1 and A_2 can be varied independently (7); i.e.,

$$D = N_{A1} + N_{A2}, \quad [31b]$$

it is possible to construct canonical decompositions of the space of solutions N_P. The corresponding theory has been developed systematically (7). Here, we recall only a few results and give some examples.

When $u \in N_P$ and $v \in N_P$, one has

$$\langle A_1u,v\rangle + \langle A_2u,v\rangle = \langle Au,v\rangle = 0. \quad [32]$$

This shows that

$$\langle A_1u,v\rangle = -\langle A_2u,v\rangle,\ \forall\ u \in N_P,\ v \in N_P. \quad [33]$$

In light of Eq. 33, one can define an operator $A_0:N_P \to N_P^*$, given by

$$\langle A_0u,v\rangle = \langle A_1,u,v\rangle = -\langle A_2u,v\rangle,\ \forall\ u \in N_P,\ v \in N_P. \quad [34]$$

Let $N_P^1 \subset N_P$ and $N_P^2 \subset N_P$ be two linear subspaces of solutions such that

(*i*) $N_P = N_P^1 + N_P^2$ [35]

(*ii*) For every $u_1 \in N_P^1$ and $v_1 \in N_P^1$, one has

$$\langle A_0u_1,v_1\rangle = \langle A_1u_1,v_1\rangle = 0. \quad [36]$$

(*iii*) For every $u_2 \in N_P^2$ and $v_2 \in N_P^2$, one has

$$\langle A_0u_2,v_2\rangle = -\langle A_2u_2,v_2\rangle = 0. \quad [37]$$

When conditions *i–iii* are satisfied, given any solution $u \in N_P$, one can write $u = u_1 + u_2$, with $u_1 \in N_P^1$ and $u_2 \in N_P^2$, because of Eq. 35. Therefore,

$$\langle A_0u,v\rangle = \langle A_0u_1,v_2\rangle + \langle A_0u_2,v_1\rangle\ \forall\ u \in N_P,\ v \in N_P, \quad [38]$$

where Eqs. 36 and 37 have been used. In view of this, it is not difficult to establish the theorem that follows.

THEOREM 2. *Given* $P:D \to D^*$ *and* A_1 *and* A_2 *satisfying Eq. 31a and b, let* $N_P^1 \subset N_P$ *and* $N_P^2 \subset N_P$ *be linear subspaces for which conditions* i–iii *hold. Define*

$$I_P = N_P + N_A \quad [39]$$

and assume that $I_P \subset D$ *is completely regular for* $A:D \to D^*$. *Then, if*

$$(N_P \cap N_{A1}) \cup (N_P \cap N_{A2}) \subset N_A, \quad [40]$$

the pair $\{N_P^1, N_P^2\}$, *constitutes a canonical decomposition of* N_P *with respect to* $A_0:N_P \to N_P^*$, *as given by Eq. 34. When this is the case,*

$$N_{A0} = I_P \cap N_A. \quad [41]$$

Proof: The proof is given in ref. 6.

We illustrate the material contained in this section by considering a simple example. Take, as in *Preliminary Results and Notation*, $D = C^\infty(R)$ and let R be the unit square, $0 < x < 1$, $0 < y < 1$. Define

$$\langle Pu,v\rangle = \int_R v\nabla^2 u\,dx + \int_0^1 u\frac{\partial v}{\partial x}\bigg|_{x=1} dy - \int_0^1 u\frac{\partial v}{\partial x}\bigg|_{x=0} dy. \quad [42]$$

Then

$$\langle Au,v\rangle = \int_0^1 \left\{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right\}\bigg|_{y=0}^{y=1} dx. \quad [43]$$

Define

$$\langle A_1u,v\} = \int_0^1 \left\{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right\}\bigg|_{y=1} dx \quad [44a]$$

$$\langle A_2u,v\rangle = -\int_0^1 \left\{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right\}\bigg|_{y=0} dx. \quad [44b]$$

Then, Eq. 31a and b is satisfied because

$$N_{A1} = \left\{u \in D \,\middle|\, u = \frac{\partial u}{\partial y} = 0 \text{ at } y = 1\right\} \quad [45a]$$

$$N_{A2} = \left\{u \in D \,\middle|\, u = \frac{\partial u}{\partial y} = 0 \text{ at } y = 0\right\}. \quad [45b]$$

Notice that the space of solutions $N_P \subset D$ is made, in this case, of the functions that are harmonic in the unit square and vanish on the vertical sides of the square; i.e.,

$$N_P = \{u \in D|\nabla^2 u = 0 \text{ on } R,\ u = 0, \text{ at } x = 0,1\}. \quad [46]$$

The space of solutions can be decomposed into two subspaces

$$N_P^1 = \{u \in N_P|u = 0 \quad \text{at} \quad y = 1\} \quad [47a]$$

$$N_P^2 = \{u \in N_P|u = 0 \quad \text{at} \quad y = 0\}. \quad [47b]$$

Then, it is straightforward to verify conditions *i–iii*.

The assumption [40] is similar to the condition that an overdetermined problem has only a trivial solution. It can also be derived, in some applications, by analytic continuation arguments. For the example given here, it follows from the fact that the only harmonic function in the square, which vanishes together with its normal derivative either at the top or at the bottom of the square, is the zero function (i.e., the function that is identically zero in the square).

In applications, $A_0:N_P \to N_P^*$ has many alternative expressions. For example, if one defines the bilinear functional $\mathcal{A}(\lambda)$ by

$$\langle \mathcal{A}(\lambda)u,v\rangle = \int_0^1 \left\{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right\}_{y=\lambda} dx;\ 0 \le \lambda \le 1 \quad [48]$$

Then, in the example considered here, for every $u \in N_P$ and $v \in N_P$, one has

$$\langle A_0u,v\rangle = \langle \mathcal{A}(\lambda)u;v\rangle,\ \forall\ \lambda \in [0,1]. \quad [49]$$

A corollary of *Theorem 2* that will be used when discussing biorthogonal functions is that, for every $u \in N_P$, one has

$$u \in N_P^1 \Leftrightarrow \langle A_0u,v\rangle = 0\ \forall\ v \in N_P^1 \quad [50a]$$

and

$$u \in N_P^2 \Leftrightarrow \langle A_0 u, v\rangle = 0 \ \forall\ u \in N_P^2. \quad [50b]$$

In the specific example given here, relationship **50a** and **b** implies that a harmonic function u that vanishes at the sides $x = 0, 1$ of the square vanishes at the top, if and only if, the integral [48] vanishes for every harmonic function v that satisfies the same conditions. Clearly, harmonic functions that vanish at the bottom of the square have a similar property.

FOURIER BIORTHOGONAL SYSTEMS

Some of the concepts presented in this section are applicable to any canonical decomposition [7–8], but here we restrict attention to canonical decompositions of the space of solutions $N_P = N_P^1 + N_P^2$.

Definition 3. A system $\{w_1, w_2, \ldots\} \subset N_P^1$ is said to be *c*-complete for N_P^1 when, for every $u \in N_P$, one has

$$\langle A_0 u, w_\alpha\rangle = 0 \ \forall\ \alpha = 1,2, \ldots \Rightarrow u \in N_P^1. \quad [51a]$$

Similarly, $\{w_1, w_2, \ldots\} \subset N_P^2$ is *c*-complete for N_P^2 when, for every $u \in N_P$,

$$\langle A_0 u, w_\alpha^*\rangle = 0 \ \forall\ \alpha = 1,2, \ldots \Rightarrow u \in N_P^2. \quad [51b]$$

Definition 4. Let $B_1 = \{w_1, w_2, \ldots\} \subset N_P^1$, $B_2 = \{w_1^*, w_2^*, \ldots\} \subset N_P^2$. Then, the systems B_1 and B_2 are said to be biorthogonal when

$$\{A_0 w_n, w_m^*\rangle = 0 \quad \text{whenever } n \neq m. \quad [52]$$

A pair of biorthogonal systems is said to be *c*-complete when B_1 is *c*-complete for N_P^1 and B_2 is *c*-complete for N_P^2. Systems B_1 and B_2 are said to be biorthonormal when

$$\langle A_0 w_n, w_m^*\rangle = \delta_{nm}. \quad [53]$$

LEMMA 1. *Assume the pair* $\mathrm{B}_1 = \{\mathrm{w}_1, \mathrm{w}_2, \ldots\} \subset \mathrm{N}_\mathrm{P}^1$ *and* $\mathrm{B}_2 = \{\mathrm{w}_1^*, \mathrm{w}_2^*, \ldots\} \subset \mathrm{N}_\mathrm{P}^2$ *is a c-complete pair of biorthogonal systems for* $\mathrm{A}_0{:}\mathrm{N}_\mathrm{P} \to \mathrm{N}_\mathrm{P}^*$ *such that*

$$\mathrm{A}_0 \mathrm{w}_\mathrm{n} \neq 0 \ \forall\ \mathrm{n} = 1,2. \quad [54]$$

Then, it can be normalized (i.e., by multiplication by a scalar of every one of its elements, one can derive a pair that is biorthonormal).

Proof: Clearly, the assertion of the lemma is true if $\langle A_0, w_n, w_n^*\rangle \neq 0$ for every $n = 1,2, \ldots$. Assume that

$$\langle A_0 w_n, w_n^*\rangle = 0 \quad [55]$$

for some n. Then,

$$\langle A_0 w_n, w_m^*\rangle = 0 \ \forall\ m = 1,2, \ldots \quad [56]$$

This implies that $w_n \in N_P^2$; i.e., $w_n \in N_P^1 \cap N_P^2 = N_{A0}$. This contradicts [54].

Notice that, when biorthogonal systems $B_1 \subset N_P^1$ and $B_2 \subset N_P^2$, which are *c*-complete, are given, with every $u \in N_P = N_P^1 + N_P^2$, one can associate unique sequences $[a_1, a_2, \ldots]$, $[b_1, b_2, \ldots]$ by means of

$$a_\alpha = \langle A_0 u, w_\alpha^*\rangle;\ b_\alpha = \langle A_0 w_\alpha, u\rangle,\ \alpha = 1,2, \ldots \quad [57]$$

Let $\mathscr{E} \subset N_P/N_{A0}$ be

$$\mathscr{E} = \left\{ u \in N_P/N_{A0} \,\middle|\, \sum_{\alpha=1}^{\infty} |a_\alpha|^2 < \infty,\ \sum_{\beta=1}^{\infty} |b_\alpha|^2 < \infty \right\}. \quad [58]$$

Then, on $\mathscr{E}$, one can define the inner product

$$(u,v) = \sum_{\alpha=1}^{\infty} a_\alpha \bar{a}'_\alpha + \sum_{\beta=1}^{\infty} b_\alpha \bar{b}'_\alpha, \quad [59]$$

where a'_α, b'_α are associated with v by equations corresponding to [57]; in addition, the bars in Eq. 59, denote the complex conjugates. Let $\mathscr{H}$ be the closure of $\mathscr{E}$ in this inner product.

Of special interest is the case of $\mathscr{H} \supset N_P/N_{A0}$. In this case, one can show that the system $B_1 \cup B_2$ is orthonormal for the Hilbert space $\mathscr{H}$, with the inner product given by Eq. 59. This inner product and the corresponding metric will be said to be induced by the biorthogonal system B_1, B_2. Notice that

$$u = \sum_{\alpha=1}^{\infty} a_\alpha w_\alpha + \sum_{\infty=1}^{\infty} b_\beta w_\beta^*, \quad [60]$$

while

$$(u,v) = \langle A_0 u, v^*\rangle. \quad [61]$$

Here,

$$v^* = -\sum_{\alpha=1}^{\infty} \bar{b}'_\alpha w_\alpha + \sum_{\alpha=1}^{\infty} \bar{a}'_\beta w_\beta^* \quad [62]$$

and convergence in Eq. 60 is with respect to the induced metric or any equivalent metric.

For applications, it is of course important to establish criteria under which the induced metric is equivalent to a metric that is relevant for the problem considered. In a previous paper (9) some aspects of this question have been discussed.

As noted in *Canonical Decompositions of the Space of Solutions*, Eq. 49, one usually has many alternative expressions for the operator $A_0{:}N_P \to N_P^*$. Let $\mathscr{A}(\lambda)$ be a family of bilinear functionals such that

$$\langle A_0 u, v\rangle = \langle \mathscr{A}(\lambda) u, v\rangle \quad [63]$$

for every $u \in N_P$ and $v \in N_P$. Consider, as before, a canonical decomposition $\{N_P^1, N_P^2\}$ of N_P. Let $w_n \in N_P^1$ and $w_n^* \in N_P^2$, $n = 1,2, \ldots$, be two families of solutions such that

$$\langle \mathscr{A}(\lambda) w_n, w_m^*\rangle = f_{nm}(\lambda) \langle \mathscr{A}(\lambda_o) w_n, w_m^*\rangle \quad [64]$$

in some range $a < \lambda < b$. Here λ_o is a fixed value belonging to this range and $f_{nm}(\lambda)$ is for every $n,m = 1,2, \ldots$ a function of λ. Then, $f_{nm}(\lambda)$ is a constant or

$$\langle A_0 w_n, w_m^*\rangle = 0. \quad [65]$$

This is a general form of Herrera's (5) alternative.

For example, let N_P be the linear space of functions that are harmonic everywhere in the plane except, possibly, the origin. Let $A_0{:}N_P \to N_P^*$ be

$$\langle A_0 u, v\rangle = \int_C \left\{ v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r} \right\} dx, \quad [66]$$

where C is any circle having center at the origin and $\partial/\partial r$ stands for the directional derivative in the radial direction. By the procedure explained above, it can be shown that a canonical decomposition of N_P is the pair $\{N_P^1, N_P^2\}$, where N_P^1 is the set of functions that are harmonic in the whole plane, including the origin and N_P^2 is made of the functions $u \in N_P$ such that $u - b_0 \log r$ is square integrable in any region of the plane that excludes a neighborhood of the origin. Here,

$$b_0 = \frac{1}{2\pi} \int_{C(\lambda)} \frac{\partial u}{\partial r}\, dx.$$

It can be seen that the only element of N_{A0} is the zero function.

A family of bilinear functionals $\mathscr{A}(\lambda)$, with property [63] is

$$\langle \mathscr{A}(\lambda) u, v\rangle = \int_{C(\lambda)} \left\{ v\frac{\partial u}{\partial r} - u\frac{\partial v}{\partial r} \right\} dx, \quad [67]$$

where $C(\lambda)$ is a circle of radius λ and center at the origin. If $w_n \in N_P^1$ and $w_n^* \in N_P^2$, $n = 1,2, \ldots$, are families of solutions of product form; i.e., if

$$w_n = f_n(r)p_n(\theta);\ w_n^* = g_n(r)q_n(\theta), \quad [68]$$

then, Eq. **67** shows that

$$\langle \mathcal{A}(\lambda)w_n, w_m^* \rangle = [g_m(\lambda)f_n'(\lambda) - f_n(\lambda)g_m'(\lambda)]\lambda\langle \mathcal{A}(1)w_n, w_m^* \rangle. \quad [69]$$

Application of the alternative (Eq. [**65**]) gives

$$\int_C \left\{ w_m^* \frac{\partial w_n}{\partial r} - w_n \frac{\partial w_m^*}{\partial r} \right\} dx = 0 \quad [70]$$

unless

$$\{g_n(\lambda)f_n'(\lambda) - f_n(\lambda)g_n'(\lambda)\}\lambda = \text{constant};\ 0 < \lambda < \infty. \quad [71]$$

Solutions of product form are

$$\{w_1, w_2, \ldots\} = \{1, r\cos\theta, r\sin\theta, \ldots\} \subset N_P^1 \quad [72a]$$

$$\{w_1^*, w_2^*, \ldots\} = \{\log r,\ r^{-1}\cos\theta, r^{-1}\sin\theta, \ldots\} \subset N_P^2. \quad [72b]$$

With these definitions, Eqs. **70** and **71** imply that

$$\langle A_0 w_n, w_m^* \rangle = 0 \quad \text{if} \quad n \neq m \text{ and } \begin{cases} n+1 \neq m \text{ when } n \text{ is even} \\ n-1 \neq m \text{ when } n \text{ is odd.} \end{cases} \quad [73]$$

This would give groups of two functions that are orthogonal to all the others. However, due to the manner in which they have been chosen, Eq. 73 holds whenever $n \neq m$.

1. Spence, D. A. (1978) *Technical Summary Report 1863* (Univ. of Wisconsin, Mathematics Research Center, Madison, WI).
2. Joseph, D. D. (1979) in *Trends in Applications of Pure Mathematics to Mechanics*, ed. Zorsky, H. (Pitman, New York), Vol. 2.
3. Smith, R. C. T. (1952) *Aust. J. Sci. Res.* **5**.
4. Alsop, L. E. (1968) *Bull. Seismol. Soc. Am.* **58**, 1949–1954.
5. Malischewsky, P. (1976) *Pure Appl. Geophys.* **114**, 833–843.
6. Herrera, I. (1964) *Bull. Seismol. Soc. Am.* **54**, 1087–1096.
7. Herrera, I. (1981) *Invest. Kinam.* **3**, 2.
8. Herrera, I. (1980) *J. Inst. Math. Its Appl.* **25**, 67–96.
9. Herrera, I. (1980) *Proc. Natl. Acad. Sci. USA* **77**, 4395–4398.