Pergamon

Int. J. Solids Structures Vol. 35, No. 33, pp. 4427–4436, 1998

0020-7683/98 $19.00 + .00

PII: S0020-7683(97)00261-8

ELIMINATION OF RIGID BODY MODES FROM DISCRETIZED BOUNDARY INTEGRAL EQUATIONS

EARLIN LUTZ
1335 Jackson Lane, West Chester, PA 19380, U.S.A.

WENJING YE and SUBRATA MUKHERJEE*
Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca NY 14853, U.S.A.

(*Received* 29 *January* 1997; *in revised form* 28 *August* 1997)

Abstract—Free rigid body modes in Neumann problems are typically eliminated by suitably restraining the body. An alternative approach, here called "regularization", involves first computing the singular stiffness matrix and then suitably modifying it using ideas from linear algebra. This idea has been suggested by Vêrchery (1990) for symmetric matrices. This paper is concerned with regularization of nonsymmetric stiffness matrices that arise from the boundary element method (BEM) for linear elasticity. Existence and uniqueness issues, as well as properties of the displacement field, for elasticity problems with tractions prescribed at every point on the boundary, are discussed in this paper.

1. INTRODUCTION

It is well known that pure Neumann problems lead to singular stiffness matrices at the discretized level. Such is the case, for example, if only tractions are prescribed on the boundary ∂B of a linearly elastic body B, or fluxes are prescribed on ∂B in potential theory. The common remedy for such problems is to apply sufficient restraint on the body by prescribing displacements (in the elasticity problem) or temperatures (in the steady state heat conduction problem) at suitable points on its boundary. An alternative approach, here called "regularization", is to first compute the singular stiffness matrix A and then modify it using ideas from linear algebra. This has been done by Vêrchery (1990) for the case where A is symmetric and positive definite. Obviously, Vêrchery's work is applicable to methods such as finite elements or symmetric Galerkin boundary elements where one obtains a symmetric positive definite stiffness matrix after discretization.

The linear algebraic approach of "regularization" of A has certain potential advantages over the commonly used "restraint" approach. Some of these are:

(1) The regularization can be applied routinely without requiring the analyst to determine how to sufficiently restrain a body for each separate problem.
(2) It is expected that the regularization approach, which is global, will lead to better numerical accuracy than the restraint approach which is often local, especially for problems without obvious regions of symmetry.

This paper is concerned with application of the regularization approach for the usual boundary element method (BEM) that leads to non-symmetric matrices. Only linear elasticity problems are considered in detail hereafter. Potential problems are easier to regularize and are sometimes alluded to in this paper.

For specifity, elasticity problems under consideration in this paper start from the boundary integral equation (BIE) first proposed by Rizzo (1967)

* Author to whom correspondence should be addressed. Tel.: 607 255 7143. Fax: 607 255 2011. E-mail: sm85@cornell.edu.

$$C_{ik}(P)u_i(P) = \int_{\partial B} [U_{ik}(P,Q)\tau_i(Q) - T_{ik}(P,Q)u_i(Q)]\,\mathrm{d}s(Q) \tag{1}$$

where $\mathbf{u}$ and $\boldsymbol{\tau}$ are boundary tractions and displacements, $\mathbf{U}$ and $\mathbf{T}$ are the usual Kelvin tensors (given in many references such as Mukherjee, 1982) and $\mathbf{C}$ is the corner tensor.

Also, P and Q are source and field points, respectively, on ∂B, and $\mathrm{d}s(Q)$ is a surface element on ∂B. Equation (1) applies to both two- and three-dimensional (2-D or 3-D) problems provided that the appropriate kernels are used.

A discretized form of eqn (1) can be written as

$$Au = B\tau \tag{2}$$

where, for specificity, let

$$[u]^T = [u_1(1), u_2(1), u_3(1), \ldots, u_1(N), u_2(N), u_3(N)] \tag{3}$$

and similarly for the traction vector. Here, 3-D problems are under consideration and N is the number of boundary nodes on ∂B. The superscript T denotes the transpose of a vector.

The square matrices A and B are, in general, nonsymmetric. The matrix A is singular with nullity k equal to six for 3-D elasticity and three for 2-D elasticity. (The corresponding discretized BIE for the Laplace equation leads to a stiffness matrix with nullity equal to 1). It is well known that the null space $\mathcal{N}(A)$ of A is spanned by the rigid body modes. Therefore, a basis of $\mathcal{N}(A)$ for 3-D elasticity problems, is

$$\begin{aligned}
\mathbf{t}_1^T &= [1,0,0,1,0,0,\ldots,1,0,0]\\
\mathbf{t}_2^T &= [0,1,0,0,1,0,\ldots,0,1,0]\\
\mathbf{t}_3^T &= [0,0,1,0,0,1,\ldots,0,0,1]\\
\boldsymbol{\rho}_1^T &= [0,-z_1,y_1,0,-z_2,y_2,\ldots,0,-z_N,y_N]\\
\boldsymbol{\rho}_2^T &= [z_1,0,-x_1,z_2,0,-x_2,\ldots,z_N,0,-x_N]\\
\boldsymbol{\rho}_3^T &= [-y_1,x_1,0,-y_2,x_2,0,\ldots,-y_N,x_N,0]
\end{aligned} \tag{4}$$

in terms of the three translation and rotation vectors $\mathbf{t}_k$ and $\boldsymbol{\rho}_k$ (each of size $3N \times 1$), respectively. An important issue is the choice of a coordinate system for specifying $\boldsymbol{\rho}_k$. This matter is addressed later in this paper.

The matrix B is nonsingular since linear elasticity guarantees a unique solution for $\boldsymbol{\tau}$ if $\mathbf{u}$ is prescribed on ∂B.

It is noted that a basis for the null space of the corresponding matrix from potential theory is a vector, of size $N \times 1$, with all its entires equal to unity.

The rest of the paper is organized as follows. First, a proposed approach for the regularization of A is presented. This is followed by a discussion of the properties of the specific unique solution that is obtained by this approach. Finally, some comments are made regarding partial regularization—i.e. when only some (but not all) of the free translation and/or rotation modes need to be eliminated in a particular boundary value problem.

2. REGULARIZATION OF THE MATRIX A

This section explains how the matrix A can be modified in order for a correct solution for $\mathbf{u}$ to exist and be unique.

2.1. *Existence of a solution* z

This proof is well known and is only repeated here for completeness.

Let $\mathbf{u}_i = \mathbf{t}_i$ for $i = 1$, 2 and 3 and $\mathbf{u}_i = \rho_i$ for $i = 4$, 5 and 6. Let $n = 3N$ and U, $n \times 6$, have the null vectors $\mathbf{u}_i$ as its columns. Now

$$Au = 0 \tag{5}$$

Any solution of a consistent linear system

$$Ax = b \tag{6}$$

can be written as

$$x = z + Uy \tag{7}$$

where the vector z is orthogonal to the null space of A, i.e.

$$u^T z = 0 \tag{8}$$

Let

$$C = A + VU^T \tag{9}$$

with V and $n \times 6$ matrix with the rank of V equal to six. Then

$$Cz = Az + VU^T z = Az + 0 = A(x - Uy) = Ax = b$$

so that one can solve

$$Cz = b \tag{10}$$

in order to find one of the solutions x of eqn (6).

2.2. *Uniqueness—a sufficient condition*

A singular value decomposition (SVD) of the matrix A can be carried out in the form

$$A = W\Sigma Z^T \tag{11}$$

with

$$W = [\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{n-6}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6]$$
$$Z = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{n-6}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_6]$$

where $\mathbf{q}_i$, $i = 1, 2, \ldots, n-6$ constitute a basis for $\mathscr{R}(A)$, the range of A, $\mathbf{v}_i$, $i = 1, 2, \ldots, 6$ is a basis for the $\mathfrak{R}^{(n)}$ complement of $\mathscr{R}(A)$, $\mathbf{p}_i$, $i = 1, 2, \ldots, n-6$ is a basis for the $\mathfrak{R}^{(n)}$ complement of $\mathscr{N}(A)$, and Σ is a diagonal matrix

$$\Sigma = \operatorname{diag}\,[\sigma_1, \sigma_2, \ldots, \sigma_{n-6}, 0, 0, 0, 0, 0, 0] \tag{12}$$

Now, let

$$C = A + VU^T \tag{13}$$

with $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6]$.

It is easy to show that the singular value decomposition of C is

$$C = W\Lambda Z^T \tag{14}$$

where

$$\Lambda = \text{diag}\,[\sigma_1, \sigma_2, \ldots, \sigma_{n-6}, 1, 1, 1, 1, 1, 1] \tag{15}$$

so that C is nonsingular and z, from equation (10), is unique for any b.

The remaining issue is an efficient way to choose V. It is necessary that rank $(V) = 6$ and it is sufficient that $\mathscr{R}(V)$ be outside of $\mathscr{R}(A)$

2.3. *Choice of* V

It is well known that for any matrix A

$$\mathscr{R}^{\perp}(A) = \mathscr{N}(A^T) \tag{16}$$

Further, for a square matrix A

$$\text{nullity}\,(A) = \text{nullity}\,(A^T) \tag{17}$$

Thus, for any square matrix A

$$\mathscr{R}(A) \bigcup \mathscr{N}(A^T) = \mathfrak{R}^{(n)} \tag{18}$$

Therefore, if A is symmetric, a sufficient choice of V, for C to be nonsingular, is

$$V = U \tag{19}$$

This has been done by Vêrchery (1990).

Unfortunately, A in eqn (2) from the BEM is not symmetric. The choice $V = U$ does not work, in general, for nonsymmetric matrices. A simple counter-example is

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for which $\mathscr{N}(A) = \mathscr{R}(A)$. It is still possible, however, to efficiently choose V such that a sufficient condition for C to be nonsingular can be met. The algorithm proposed for this purpose is given below.

First, note that in order for a solution $\mathbf{u}$ of eqn (2) to exist, $\boldsymbol{\tau}$ must satisfy equilibrium, i.e., the discretized versions of the equations

$$\int_{\partial B} \boldsymbol{\tau}\, \mathrm{d}s = 0 \tag{20}$$

$$\int_{\partial B} (\mathbf{r} \times \boldsymbol{\tau})\, \mathrm{d}s = 0 \tag{21}$$

where $\mathbf{r}$ is the radius vector from any point to a boundary point.

The proposed algorithm for choosing V is

1. Choose six linearly independent traction vectors, $\boldsymbol{\tau}_i$, $i = 1, 2, \ldots, 6$ that violate equilibrium. This guarantees that for any of these choices, a solution u from eqn (2), does not exist. Therefore, the vectors $B\boldsymbol{\tau}_i$ are outside the range of A.

A simple choice for τ_i is

$$\tau_i = \mathbf{t}_i, \quad i = 1, 2, 3$$
$$\tau_i = \boldsymbol{\rho}_i, \quad i = 4, 5, 6 \tag{22}$$

The vectors $\boldsymbol{\rho}_i$ can be evaluated in an arbitrary coordinate system. The reasons for this are explained later in the paper.

2. Since B is nonsingular (by virtue of the uniqueness theorem of linear elasticity)

$$\mathbf{d}_i = B\tau_i, \quad i = 1, 2, \ldots, 6 \tag{23}$$

are linearly independent.

3. The vectors $\mathbf{d}_i$ are outside $\mathscr{R}(A)$ by virtue of statements (1) and (2) above.
Set

$$V = [\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_6] \tag{24}$$

2.4. *Scaling of equations*

In practice, when using eqn (10), it is important to scale the matrices U and V such that the entries of VU^T are of the same order as the matrix A.

3. PROPERTIES OF THE UNIQUE SOLUTION

This section discusses additional properties of the unique solution. First, the unique displacement solution is shown to satisfy six conditions that are in the form usually associated with equilibrium conditions for forces. Second, for a given particular solution composed of the unique solution with added rigid body modes there is a direct way to remove the rigid body parts and return to the unique part. Both of these properties have both discrete (matrix) and continuum (integral) formulae.

Let $u^{(0)}$ be the (unique) solution of the regularized BEM equation

$$Cu^{(0)} = B\tau \tag{25}$$

From (8), we know that $U^T u^{(0)} = \mathbf{0}$, where the $\mathbf{0}$ is a 6-dimensional zero vector. Each of the first three components has the form $t_i^T u^{(0)} = 0$, where $\mathbf{t}_i$ is a pure translation vector, and is the summation of a particular displacement component at all the nodes. These summations are a discrete analog of the continuum vector integral

$$\int_{\partial B} \mathbf{u}\, \mathrm{d}s = 0 \tag{26}$$

The last three components of $U^T u^{(0)} = \mathbf{0}$ are of the form $\rho_i^T u^{(0)} = 0$ with rotation $\boldsymbol{\rho}_i$. These summations are the discrete analog of the continuum vector integral

$$\int_{\partial B} (\mathbf{r} \times \mathbf{u})\, \mathrm{d}s = 0 \tag{27}$$

Equations (26) and (27) are of precisely the same form as the force and moment equilibrium eqns (20) and (21), but with displacement taking the place of traction! Also, the "force equilibrium" condition (26) allows the "momentum equilibrium" condition to be measured from an arbitrary point. This is analogous to the fact that for a body in equilibrium, the sum of the moments of all the applied forces about a point is 0, regardless of the location of that point.

From (7), it is known that any other $u^{(p)}$ that is a particular solution of $Au^{(p)} = B\tau$ differs for the solution $u^{(0)}$ by a linear combination of the (rigid body motion) columns of U, i.e.

$$\mathbf{u}^{(p)} = \mathbf{u}^{(0)} + [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3] \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} + [\boldsymbol{\rho}_1 \quad \boldsymbol{\rho}_2 \quad \boldsymbol{\rho}_3] \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{28}$$

where the multipliers for the linear combination of columns have interpretations as translations δ_1, δ_2, δ_3 and rotations ω_1, ω_2, ω_3. This interpretation applies to both the $3N$ dimensional vector solution (28) and on a point-by-point basis in Cartesian space. That is, by expanding the rigid body vectors as defined in (4), one obtains the displacement components for $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(p)}$ at any particular node as

$$\mathbf{u}^{(p)} = \mathbf{u}^{(0)} + \boldsymbol{\delta} + \boldsymbol{\omega} \times \mathbf{r} \tag{29}$$

If the rotations are taken about the centroid of the nodal points, and one uses principal coordinates, one has

$$t_i^T t_j = \delta_{ij} N, \quad t_i^T \rho_j = 0, \quad \rho_i^T \rho_j = \begin{cases} 0 & \text{for } i \neq j \\ I_i, & \text{for } i = j \end{cases} \tag{30}$$

where I_i is the second moment of the nodes about the x_i axis (e.g. $I_1 = \Sigma_k (y_k^2 + z_k^2)$ and x_k, y_k, z_k are (principal centroidal) nodal coordinates of node k). Please note that the principal centroidal coordinate system is determined by appropriate summations of the discrete nodal coordinates.

In principal centroidal coordinates, one can extract δ_1, δ_2, δ_3 and ω_1, ω_2, ω_3 from a given u^p by taking inner products of both sides of (28) with the rigid body vectors and applying (30) and $U^T u^{(0)} = 0$:

$$\delta_i = \frac{1}{N} t_i^T u^{(p)}, \quad \omega_i = \frac{1}{I_i} \rho_i^T u^{(p)}, \quad i = 1, 2, 3 \tag{31}$$

Once these are known, $\mathbf{u}^{(0)}$ can be obtained directly by simple rearrangement of (28). Clearly, the same $\mathbf{u}^{(0)}$ is obtained regardless of which particular $\mathbf{u}^p$ is chosen for the 6-dimensional rigid body motion space.

As was done for (26) and (27), if one interprets the inner products in (31) as discrete analogs of continuum integrals, one obtains formulae to extract translation and rotation parts from a particular displacement solution field $\mathbf{u}^p$ as

$$\delta_i = \frac{1}{A} \int_{\partial B} u_i^{(p)} \, \mathrm{d}s, \quad \omega_i = \frac{1}{I_i} \int_{\partial B} [\mathbf{r} \times \mathbf{u}]_i \, \mathrm{d}s \tag{32}$$

where A is the surface area of the body.

Finally, one gets

$$\mathbf{u}^{(0)}(\mathbf{x}) = \mathbf{u}^{(p)}(\mathbf{x}) - \frac{1}{A} \int_{\partial B} \mathbf{u}^{(p)} \, \mathrm{d}s - \frac{1}{I_k} \left\{ \int_{\partial B} [\mathbf{r} \times \mathbf{u}^{(p)}]_k \, \mathrm{d}s \right\} \mathbf{e}_k \times \mathbf{r} \tag{33}$$

If the translations and rotations are expressed in other coordinate systems (e.g. not centroidal or not in principal orientation), the δ_i and ω_i can be obtained by solving the (6×6) linear system of the form

$$U^T u^{(p)} = U^T U \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{34}$$

The choice of centroidal coordinates simplifies the problem by making $U^T U$ diagonal. (Scaling appropriate columns of U by $\sqrt{N}$ and $\sqrt{I_i}$ simplifies the solution even further, making $U^T U$ an identity matrix.)

In two dimensions, one has three linearly independent vectors in the null space of A.

$$\begin{aligned} \mathbf{t}_1^T &= [1, 0, 1, 0, \ldots, 1, 0] \\ \mathbf{t}_2^T &= [0, 1, 0, 1, \ldots, 0, 1] \\ \boldsymbol{\rho}^T &= [-y_1, x_1, -y_2, x_2, \ldots, -y_n, x_n] \end{aligned} \tag{35}$$

Now one can follow the same procedure as before to obtain, in centroidal coordinates,

$$\delta_i = \frac{1}{N} t_i^T u^{(p)}, \quad i = 1, 2 \tag{36}$$

$$\omega = \frac{1}{I} \rho^T u^{(p)} \tag{37}$$

where I is the second moment of the boundary nodes about an axis normal to the body through its centroid, i.e.

$$I = \sum_k (x_k^2 + y_k^2) \tag{38}$$

The continuum analog of these equations is

$$\mathbf{u}^{(0)}(\mathbf{x}) = \mathbf{u}^{(p)}(\mathbf{x}) - \frac{1}{L} \int_{\partial B} \mathbf{u}^{(p)} \, \mathrm{d}s - \frac{1}{I} \left[\int_{\partial B} [\mathbf{r} \times \mathbf{u}^{(p)}]_3 \, \mathrm{d}s \right] \mathbf{e}_3 \times \mathbf{r} \tag{39}$$

where L is the perimeter of the body.

The regularization approach presented in this paper calculates the elastic deformation and stress fields with respect to a reference rigid body configuration that satisfies the constraint eqns (26)–(27). It is important to mention here that for bodies moving freely, such as space structures, this reference configuration is of little practical consequence, provided that $\mathbf{u}^{(0)}$ from eqn (28) is unique. The calculated stresses in the body, of course, are also unique. Also, it is restated for emphasis that, in practical applications, the entire problem including the evaluation of $\boldsymbol{\rho}_k$, can be carried out in arbitrary coordinates.

4. A 2-D EXAMPLE WITH UNIFORM STRAINS

Consider a 2-D body of arbitrary shape, in plane strain to be specific, subjected to spatially uniform normal strains ε_{11} and ε_{22} with zero shear ($\varepsilon_{12} = 0$). The general form of a particular solution $\mathbf{u}^{(p)}$, in arbitrary coordinates, is

$$u_1^{(p)} = \varepsilon_{11}x + \omega_{12}y + E_1$$
$$u_2^{(p)} = \varepsilon_{22}y - \omega_{12}x + E_2 \tag{40}$$

where ω_{12} is an arbitrary rotation and $\mathbf{E}$ is an arbitrary translation.

A simple solution for $\mathbf{u}^{(0)}$ is obtained from eqn (39) if centroidal principal coordinates $\hat{\mathbf{x}}$ are used. This is

$$\mathbf{u}^{(0)}(\hat{\mathbf{x}}) = \begin{pmatrix} \varepsilon_{11}\hat{x} \\ \varepsilon_{22}\hat{y} \end{pmatrix} \tag{41}$$

This means that if the problem [including the rotation vector $\boldsymbol{\rho}$ in eqn (35)], is formulated in centroidal principal coordinates, the computed unique solution is such that the initial configuration does not undergo any rigid body motion. Again, it should be emphasized that it is not necessary in practice to use any special coordinate system. Any arbitrary coordinates would suffice in order to get a unique solution $\mathbf{u}^{(0)}$. The purpose of the exercise in this section is to understand the nature of the unique solution in this special case with uniform normal strains.

A numerical example is shown in Fig. 1. An irregular hexagon, in plane strain, is subjected to a boundary traction field corresponding to equal biaxial tension, i.e.

$$\sigma_{11} = 1, \quad \sigma_{22} = 1, \quad \sigma_{12} = 0$$

which gives

$$\boldsymbol{\tau} = \mathbf{n}$$

where $\mathbf{n}$ is the unit outward normal to ∂B.

The discretized form of a BEM elasticity code from Becker (1992) was regularized and used to solve this problem. The Young's modulus and Poisson's ratio were taken as 1 and 0.3, respectively. It is observed from the deformed shape of the body in Fig. 1 that it

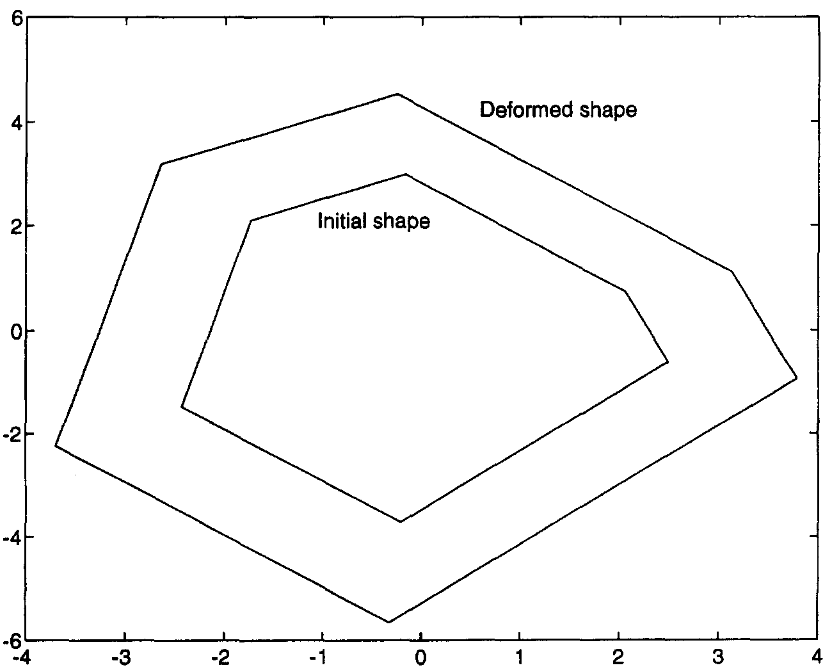

Fig. 1. A polygonal body subjected to boundary tractions corresponding to equal biaxial tension.

experiences pure expansion without any rigid body motion. Of course, the computed displacements were verified against the closed form analytical solution from eqn (41). The numerical and analytical results agree within about 1%.

5. PARTIALLY RESTRAINED BODIES

It is common to face problems in which some but not all the rigid body modes are restrained. Some simple 2-D examples are shown in Fig. 2. Figure 2(a) shows a body with a free translation mode in the y direction, while Fig. 2(b) shows a body with a free rotation about the z axis.

The first class of problems are easy to regularize with the present approach. Thus, for example, for Fig. 2(a), one uses

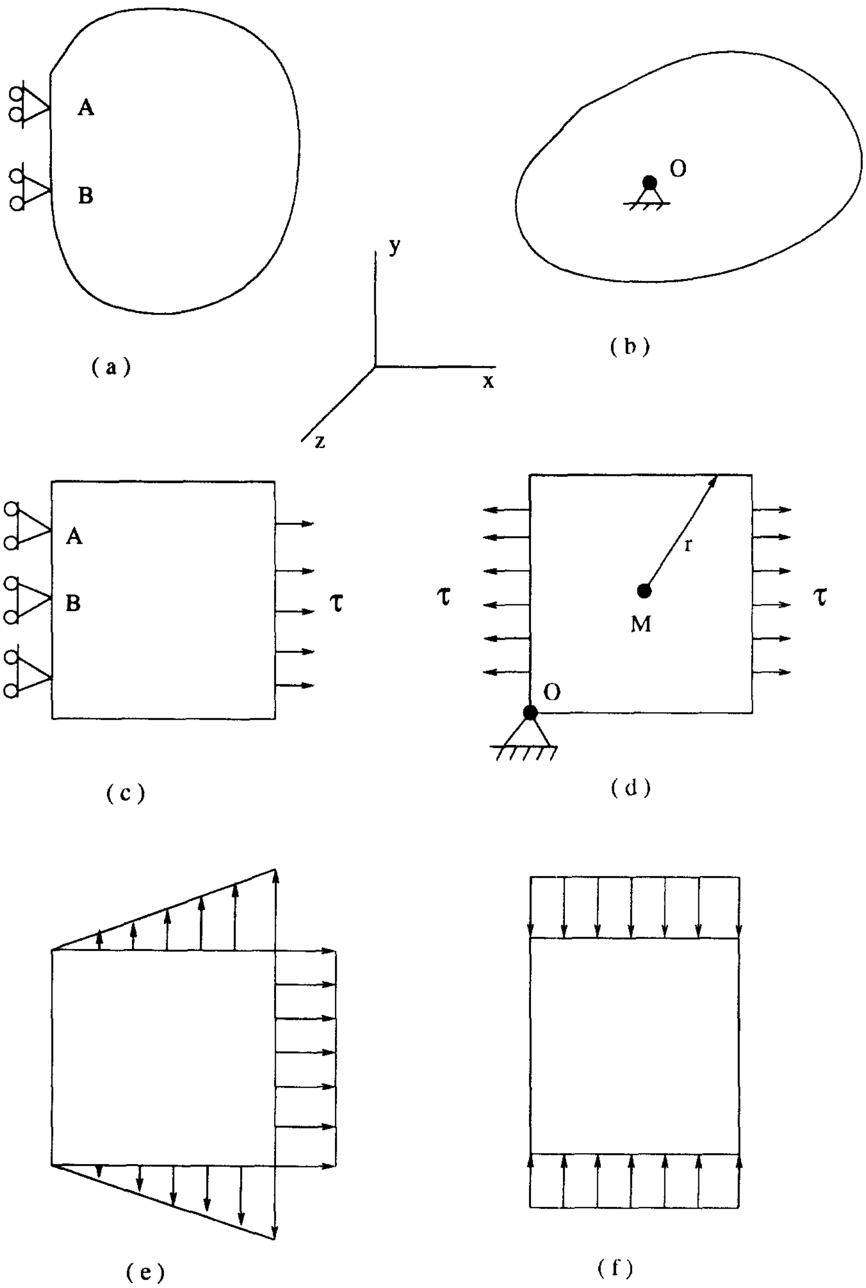

Fig. 2. Partially restrained bodies.

$$\mathbf{t}_2^T = \mathbf{U}^T = [0, 1, 0, 1, \ldots, 0, 1]^T$$

with V computed from U in the usual way. An example problem, 2(c), was solved with this approach and, as expected, a correct solution was obtained.

The situation however, is different, in general, for problems with free rotations. Of course, one would use

$$\boldsymbol{\rho}^T = \mathbf{U}^T = [-y_1, x_1, -y_2, x_2, \ldots, -y_N, x_N]$$

The difficulty that one now faces, however, is that eqn (26), "displacement equilibrium", in general, is not satisfied any more since an arbitrary point O in B might be fixed. One is now faced with choosing a suitable point M in B such that, with $\mathbf{r}$ measured from M, eqn (27) is satisfied. This is analogous to a situation in a mechanics problem where one tries to find a point about which a system of forces, acting on a body, gives zero resultant moment even though the forces themselves are not in equilibrium! Of course, for a simple example such as the one in Fig. 2(d), M is such a point. Sure enough, defining $\boldsymbol{\rho}$ with M as the origin gives a correct solution with the u_1 and u_2 profiles shown in Fig. 2(e) and 2(f), respectively. Note that this displacement profile violates eqn (26) while satisfying eqn (27) as long as $\mathbf{r}$ is measured from the point M! In general, however, it is not easy to find M so that it would be difficult to apply the present approach to problems such as in Fig. 2(b).

6. CONCLUSIONS

The regularization method, presented in this paper, is an attractive alternative to the usual one of suitably restraining a body in order to eliminate rigid body modes in Neumann problems. It is shown that previous work on regularizing symmetric stiffness matrices does not apply, in general, to nonsymmetric ones such as those that arise from the BEM. A new method is presented for regularizing nonsymmetric stiffness matrices. It is interesting to observe that the unique displacement field, obtained from this approach, satisfies equations that have exactly the same forms as the standard equations for force and moment equilibrium!

REFERENCES

Becker, A. A. (1992) *The Boundary Element Method in Engineering*. McGraw-Hill Book Company, London.
Mukherjee, S. (1982) *Boundary Element Methods in Creep and Fracture*. Elsevier Applied Science, London.
Rizzo, F. J. (1967) An integral equation approach to boundary value problems of classical elastostatics. *Qly. Appl. Math.* **25,** 83–95.
Vêrchery, G. (1990) Régularisation du système de l'équilibre des structures élastiques discrètes. *C. R. Acad. Sci. Paris*, **311,** Série II, 585–589.