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Hysteretic damping revisited

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An integral–differential equation (IDE) in the time domain has been proposed for the free vibration of a single-degree-of-freedom (SDOF) system with hysteretic damping which is different from the conventional complex stiffness model as employed in the frequency domain (Chen & You, Proposal NSC 85-2211-E-019-004, National Taiwan Ocean University, 1996; *Proc. 3rd National Congress Structural Engineering*, 1996; *BETECH96 Conf.*, 1996).[1–3] The integral of the Hilbert transform is embedded in the IDE and is calculated in the Cauchy principal value sense in Refs 1–3. In this paper, we extend the SDOF system to multi-degrees-of-freedom systems and continuous systems. Also, the ratio of dissipation energy vs time for SDOF systems is constructed for the free vibration and compared with viscous damping. Numerical examples for different loss factors have been presented and the lack of dependence of the exciting frequency on the dissipation energy has been confirmed when the maximum responses are the same. © 1997 Elsevier Science Limited. All rights reserved.

Key words: integral–differential equation, hysteretic damping, time-domain approach, Hilbert transform, causality and dissipation energy.

1 INTRODUCTION

In 1994, Chen *et al.*[4] successfully solved the SDOF hysteretic damping model in the time domain by using the concept of a phase plane. Although the model in Ref. 4 satisfies the causal effect, Crandall[5] criticized the model in Ref. 4 for not being fully equivalent to the hysteretic damping model in the frequency domain. By taking the Fourier transform with respect to the model in Ref. 4, we cannot obtain the complex stiffness of $k(1 + i\eta)$. The above statement can prove that Crandall's comments in Ref. 5 are right. Also, this finding stimulates the research on the time-domain formulation for the hysteretic damping in 1994.[1] It is interesting that Chen's study[1–3] and Inaudi's work[6] both derive an integral–differential equation (IDE) in the time-domain approach. For a long time, the damping characteristic was often utilized to suppress the vibration level using various dissipation mechanisms. In this decade, two books on the topic of damping have been published.[7,8] However, a great deal of effort has been focused on the frequency-domain approach, especially for the hysteretic damping model instead of the time-domain approach.

In this paper, we employ a direct iteration technique to solve the time-domain governing equation for harmonic loading. The hysteresis loop is constructed by using the time-domain approach. The transient behavior from origin to steady state on the damping ellipse is found. Also, the curve of the ratio of energy dissipation to time for the free vibration of hysteretic damping is constructed and is compared with that of viscous damping. To prove that the present model in the time domain is fully equivalent to that in the frequency domain, the relation between dissipation energy and exciting frequency is considered. In order to make the present formulation more practical, the extensions to multiple degrees of freedom and continuous systems are shown to see its validity.

2 FORMULATION

The governing equation of a single-degree-of-freedom (SDOF) system for the hysteretic damping model has been formulated as:[9]

$$m\ddot{u} + \frac{h}{\omega}\dot{u} + ku = \bar{p}(\bar{\omega})e^{i\bar{\omega}t} \tag{1}$$

where m, h and k represent the mass, hysteretic damping coefficient and stiffness, respectively, and $\bar{p}$ and $\bar{\omega}$ are the amplitude of the harmonic loading and the exciting frequency, respectively. To make the transfer functions conjugate for $-\bar{\omega}$ and $\bar{\omega}$, the governing equation has

been modified to be:[4]

$$m\ddot{u} + \frac{h}{|\bar{\omega}|}\dot{u} + ku = \bar{p}(\bar{\omega})e^{i\bar{\omega}t} \qquad (2)$$

Although good for harmonic motion, eqn (2) is invalid for free vibration since, when the forcing term, $\bar{p}e^{i\bar{\omega}t}$, is set to vanish, the presence of $|\bar{\omega}|$ in the denominator of eqn (2) is ambiguous. Therefore, only the steady-state solution can be obtained. In other words, the hysteretic damping is focused on the frequency domain model[10,11] as follows:

$$-m\bar{\omega}^2\bar{x} + k(1 + \text{sgn}(\bar{\omega})i\eta)\bar{x} = \bar{p} \qquad (3)$$

where η is the loss factor, $k\eta$ is equal to h, and $\text{sgn}(\bar{\omega})$ is $+1$ when $\bar{\omega} > 0$ and -1 when $\bar{\omega} < 0$. To solve for the time-domain response, two approaches in the frequency domain[2,3] have been employed as follows:

(1) Method using a real integral for $\bar{p} = 1$

$$x(t) = \frac{1}{\pi}\int_0^\infty \frac{(k - m\bar{\omega}^2)\cos\bar{\omega}t + k\eta\sin\bar{\omega}t}{(k - m\bar{\omega}^2)^2 + (k\eta)^2}d\bar{\omega} \qquad (4)$$

(2) Method using the FFT technique for $\bar{p} = 1$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{i\bar{\omega}t}}{(k - m\bar{\omega}^2 + \text{sgn}(\bar{\omega})ik\eta)}d\bar{\omega} \qquad (5)$$

where sgn is the sign function. By taking the inverse Fourier transform of eqn (3), the governing equation in the time domain, which has been derived by Chen & You[1-3] and Inaudi & Kelly[6] independently, is as follows:

$$m\ddot{x}(t) - \frac{k\eta}{\pi}\int_{-\infty}^\infty \frac{x(u)}{(t-u)}du + kx(t) = p(t) \qquad (6)$$

with the conditions at $t = -\infty$ being

$$x(t)|_{t=-\infty} = 0, \qquad \dot{x}(t)|_{t=-\infty} = 0 \qquad (7)$$

Equations (6) and (7) can be viewed as the governing equation and initial conditions in the time domain for the hysteretic damping model, respectively. If external excitation, $p(t)$, in eqn (6) vanishes, the solution, $x(t)$, becomes the free vibration, which has been found[2,3] to be equal to the solution obtained by using the method of a real integral in eqn (4) or using FFT in eqn (5). To construct the hysteresis loop in the time domain, the forcing function, $p(t)$, in eqn (6) must be harmonic excitation. In order to solve the integral–differential equation for eqn (6) by using the iteration technique, eqn (6) can be reduced to the following form:

$$\ddot{x}_{n+1}(t) + 2\xi\omega\dot{x}_{n+1}(t) + \omega^2 x_{n+1}(t) = 2\xi\omega\dot{x}_n(t)$$
$$+ \frac{\omega^2\eta}{\pi}\int_{-\infty}^\infty \frac{x_n(u)}{(t-u)}du + \frac{p(t)}{m} \qquad (8)$$

where $x_n(t)$ denotes the nth iteration state for $x(t)$, ξ is the artificial viscous damping ratio, and $\omega = \sqrt{k/m}$. By

using the Duhamel integral and treating the terms on the right-hand side of the equals sign in eqn (8) as external forces, eqn (8) can be reduced to the following iteration form:

$$x_{n+1}(t) = \frac{1}{\omega\sqrt{1 - \xi^2}}\int_0^t e^{-\xi\omega(t-\tau)}$$
$$\times \sin\left(\omega\sqrt{1 - \xi^2}(t - \tau)\right) \qquad (9)$$
$$\times \left\{\frac{\omega^2\eta}{\pi}\int_{-\infty}^\infty \frac{x_n(u)}{(\tau - u)}du + 2\xi\omega x_n(\tau) + \frac{p(\tau)}{m}\right\}d\tau$$

By iterating $x_n(t)$ in eqn (9), a hysteresis loop can be constructed after setting harmonic loading for $p(t)$, and the convergent solution can be obtained using the following criterion:

$$\int_{-\infty}^\infty |x_n(t) - x_{n+1}(t)|^2 dt < \epsilon, \qquad \text{if } n > N \qquad (10)$$

where ϵ is the error tolerance and N is the number of iterations. It is noted that the Hilbert transform is present in the damping force term of eqn (6) and in the forcing term of eqn (9), so an integral in the Cauchy principal value sense must be considered as in Refs 2 & 3.

3 NUMERICAL RESULTS AND DISCUSSION

3.1. Ratio of energy dissipation curve for free vibration

By setting

$$m = 1\,\text{kg}, \qquad k = 4\pi^2\,\text{N/m}, \qquad \omega = 2\pi\,\text{rad/s}$$

the damping force, $F_d(t)$, at time t can be determined by

$$F_d(t) = -\frac{k\eta}{\pi}\int_{-\infty}^\infty \frac{x(u)}{t - u}du$$

where $x(t)$ is the displacement history of free vibration. The ratio of the energy dissipation at time t is defined by

$$\pi_D = \frac{\int_0^t F_d(u)\dot{x}(u)du}{\int_0^\infty F_d(u)\dot{x}(u)du}$$

The ratio of the dissipation energy curve for the free vibration is shown in Fig. 1(a, b) for hysteretic damping and viscous damping, respectively. It is found that the total energy is damped out as time approaches infinity for both cases. Also, the time to damp out 95% of the total energy is shown in Fig. 1(a, b).

3.2 Hysteresis loop using the time-domain approach

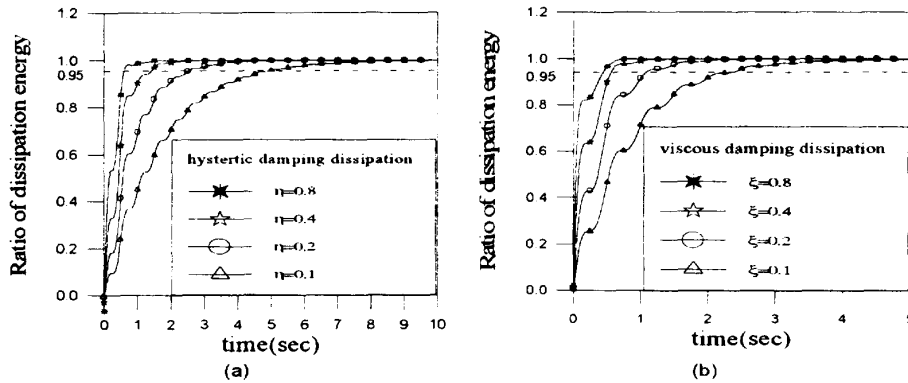The hysteresis loop is solved in the time domain by

Fig. 1. Ratio of dissipation energy vs time for (a) hysteretic damping and (b) viscous damping.

setting the forcing function, $p(t)$, in eqn (9), to be a harmonic loading: $\sin 2\pi(H(t - 20) - H(t - 30))$ for $20 < t < 30\,\mathrm{s}$, where $H(t)$ is the Heaviside function as shown in Fig. 2(a). The convergent solution can be obtained in six iterations ($N = 6$) for the error tolerance $\epsilon = 1 \times 10^{-6}$ when the maximum response of $x(t)$ is $0 \cdot 03$ for eqn (9) when the artificial damping ratio is chosen to be $0 \cdot 8$. The steady-state output occurs in the second cycle after the transient behavior as shown in Fig. 2(b). The transient behavior in the development to steady-state response on the damping ellipse is shown in Fig. 3.

The curve originates from the second quadrant since a noncausal effect is present not only in the free vibration but also in the forced vibration as shown in Fig. 2(b) for nonzero response at $t < 20\,\mathrm{s}$. It can be seen that the ellipse for the steady-state response matches the result obtained by using the frequency-domain approach very well.

3.3 The relationship between dissipation energy and loss factor

To demonstrate that the present model in the time domain is fully equivalent to that in the frequency

domain, four examples for different loss factors, $\eta = 0 \cdot 1$, $0 \cdot 2$, $0 \cdot 4$ and $0 \cdot 8$, are presented to demonstrate the relationship between the dissipation energy and strain energy. Table 1 shows that the dissipation energy, W, satisfies

$$W = 2\pi V\eta$$

where V is the maximum strain energy, as shown below:

$$V = \frac{1}{2}kX_{\max}^2$$

in which $X_{\max}$ is the maximum response. This result matches the definition of hysteretic damping in the frequency domain approach.

3.4 The relationship between dissipation energy and exciting frequency

In order to understand that the dissipation energy for hysteretic damping is independent of exciting frequencies, Table 2 shows that the dissipation energy for the SDOF hysteretic system subjected to the four different frequencies, $\bar{\omega} = 0 \cdot 5\pi$, π, $1 \cdot 5\pi$ and $2 \cdot 0\pi$, is the same when the maximum responses, $X_{\max}$, are the same.
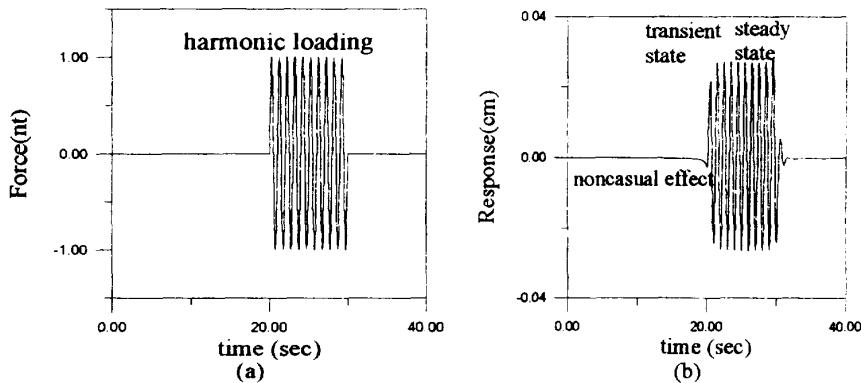

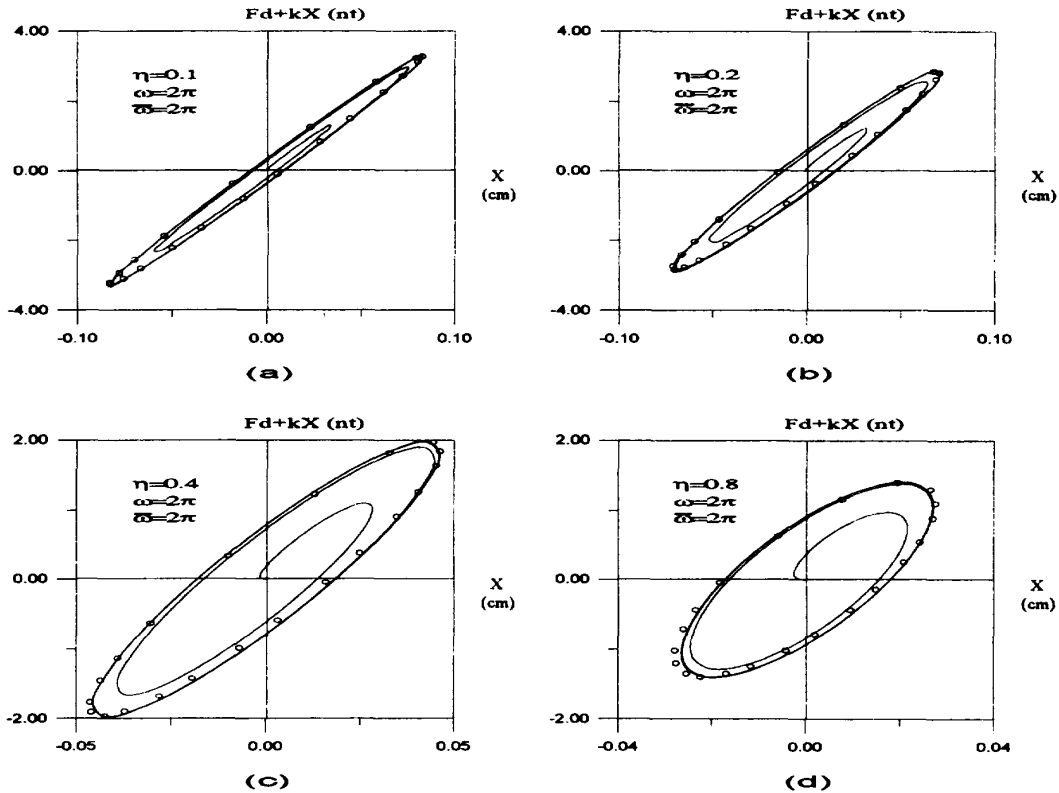**Fig. 2.** (a) Forcing function, (b) response function.

Fig. 3. Hysteresis loop obtained by using the time-domain approach and the frequency-domain approach: (a) $\eta = 0\cdot1$, (b) $\eta = 0\cdot2$, (c) $\eta = 0\cdot4$, (d) $\eta = 0\cdot8$. ——— time domain, o o o frequency domain.

3.5 Extension to multiple-degree-of-freedom (MDOF) systems

The proposed technique can be utilized in the case of MDOF systems with linear hysteretic damping. Consider the MDOF system with the governing equation

$$M\ddot{x} - K\eta\tilde{x} + Kx = F \tag{11}$$

where

$$\tilde{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(u)}{(t-u)}\,du \tag{12}$$

and x is the displacement vector, M, K and F denote the matrix, the stiffness matrix and the load vector, respectively.

By using the modal reaction method,[12,13] eqn (11) can be reformulated to an integral–differential equation of

SDOF as follows:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) - \eta\omega_i^2\tilde{q}_i(t) = \phi_i^T F(t), \qquad i = 1, 2, \ldots N \tag{13}$$

where ω_i denotes the ith natural frequency and ϕ_i^T is the transpose of the ith mode.

For simplicity, a two-DOF system with proportional damping is considered as shown in Fig. 4 with the following system parameters,

$$m_1 = 200\,\text{Mg}, \qquad \eta_1 = 0\cdot1, \qquad k_1 = 150\,\text{MN/m}$$

$$m_2 = 250\,\text{Mg}, \qquad \eta_2 = 0\cdot1, \qquad k_2 = 75\,\text{MN/m}$$

and the external force vector

$$F_1(t) = 2500\sin\pi t[H(t-2) - H(t-8)],$$

$$F_2(t) = 0$$

where $H(t)$ is the Heaviside function.

Table 1. Dissipation energy for different loss factors with $p(t) = \sin 2\pi t(H(t-20) - H(t-30))$

	$\eta = 0\cdot1$	$\eta = 0\cdot2$	$\eta = 0\cdot4$	$\eta = 0.8$
Max. response (X_{max})	0·0865	0·074	0·0487	0·0292
Strain energy ($V = \frac{1}{2}kX_{max}^2$)	0·1478	0·1104	0·0468	0·01682
Area of ellipse, dissipation energy (W)	0·0939	0·1389	0·1179	0·0846
Loss factor ($\eta = \frac{W}{2\pi V}$)	0·1	0·2	0·4	0·8

Table 2. Dissipation energy for the SDOF hysteretic system subjected to four different excitation frequencies, $\bar{\omega} = 0.5\pi$, $\bar{\omega} = \pi$, $\bar{\omega} = 1.5\pi$, $\bar{\omega} = 2\pi$, for $\eta = 0.8$ and $p(t) = A\sin\bar{\omega}t(H(t-10) - H(t-40))$

	$\bar{\omega} = 0.5\pi$	$\bar{\omega} = \pi$	$\bar{\omega} = 1.5\pi$	$\bar{\omega} = 2\pi$
Amplitude (A)	45·25	41·49	34·48	34·25
Max. response (X_{max})	1	1	1	1
Strain energy $(V = \frac{1}{2}kX_{max}^2)$	19·74	19·74	19·74	19·74
Area of ellipse, dissipation energy (W)	99·22	99·22	99·22	99·22
Loss factor $(\eta = \frac{W}{2\pi V})$	0·8	0·8	0·8	0·8

The governing equation for the MDOF system in eqn (11) is reduced to

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1\eta_1 + k_2\eta_2 & -k_2\eta_2 \\ -k_2\eta_2 & k_2\eta_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

where the explicit forms of M, C, K and F matrices are shown below:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 200 & 0 \\ 0 & 250 \end{bmatrix} \tag{14}$$

$$C = \begin{bmatrix} k_1\eta_1 & -k_1\eta_1 \\ -k_1\eta_1 & k_1\eta_1 + k_2\eta_2 \end{bmatrix} = \begin{bmatrix} 15 & -15 \\ -15 & 22.5 \end{bmatrix} \tag{15}$$

$$K = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} = \begin{bmatrix} 150 & -150 \\ -150 & 225 \end{bmatrix} \tag{16}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix} \tag{17}$$

Equation (13) for the two generalized coordinates, $q_1(t)$ and $q_2(t)$, are decoupled as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} - \begin{bmatrix} 15 & 0 \\ 0 & 150 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} 150 & 0 \\ 0 & 1500 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1.6667 \times 10^{-3} \\ 1.4907 \times 10^{-3} \end{bmatrix} F_1(t)$$

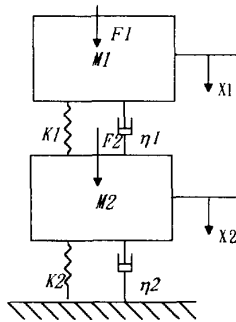By employing the same techniques as for SDOF systems the responses for the two DOFs are shown in Fig. 5.

Fig. 4. A two-DOF system.

3.6 Extension to continuous systems

Consider a shear beam model as shown in Fig. 6 with the following governing equation:

$$-G\frac{\partial^2 u(x,t)}{\partial x^2} + G\eta\frac{\partial^2 \tilde{u}(x,t)}{\partial x^2} + \rho(x)\ddot{u}(x,t) = -p(x,t),$$
$$0 < x < l \tag{18}$$

where

$$\tilde{u}(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(x,\tau)}{(t-\tau)}d\tau$$

and ρ is density, $u(x,t)$ denotes the displacement at position x and time t, '·' denotes the differentiation with respect to time, G is the shear modulus and $p(x,t)$ is the external force. For simplicity, it is assumed to be zero.

The boundary conditions are

$$u(0,t) = a(t), \qquad u_x(l,t) = 0 \tag{19}$$

where $a(t)$ is support motion. Assuming that the motion starts from rest, the initial conditions are

$$u(x,0) = 0, \qquad \dot{u}(x,0) = 0 \tag{20}$$

The solution can be decomposed as $u(xt) = a(t) + \sum_{n=1}^{\infty} q_n(t)u_n(x)$ where $u_n(x)$ is the mode shape and $q_n(t)$ is the generalized coordinate.

By using the modal reaction method,[12,13,15] eqn (18) can be reformulated as

$$\ddot{q}_n(t) + \omega_n^2 q_{n+1}(t) - \frac{\omega_n^2\eta}{\pi}\int_{-\infty}^{\infty}\frac{q_n(\tau)}{(t-\tau)}d\tau = \frac{\ddot{f}_n(t)}{N_n},$$
$$n = 1,2,3\ldots \tag{21}$$

where

$$N_n = \rho\frac{l}{2}$$

$$f_n(t) = \frac{-G(2n-1)\pi}{2\omega_n^2 l}a(t)$$

$$\omega_n = \frac{(2n-1)\pi}{2l}\sqrt{G/\rho}$$

By introducing the artificial damping for iteration scheme, eqn (21) becomes

$$\ddot{q}_n^{j+1}(t) + 2\xi\omega_n\dot{q}_n^{j+1}(t) + \omega_n^2 q_n^{j+1}(t)$$
$$= 2\xi\omega_n\dot{q}_n^j(t) + \frac{\omega_n^2\eta}{\pi}\int_{-\infty}^{\infty}\frac{q_n^j(\tau)}{(t-\tau)}d\tau + \frac{\ddot{f}_n(t)}{N_n} \tag{22}$$

where $q_n^{j+1}(t)$ denotes the $(j + 1)$th iteration solution for the nth generalized coordinate, $q_n(t)$, and ξ is artificial damping. After we solve the response, the damping force can be obtained by

$$F_d(x, t) = G\eta \frac{\partial \tilde{u}(x, t)}{\partial x}, \qquad 0 < x < l \qquad (23)$$

$$= G\eta \sum_{n=1}^{\infty} \tilde{q}_n(t) \frac{\partial u_n(x)}{\partial x}, \qquad (24)$$

Consider that the shear beam has the following properties:

$$G = 1\,\text{N/m}^2, \quad \eta = 0.8, \quad \rho = 1\,\text{kg/m}^3 \quad \text{and} \quad l = 1\,\text{m}$$

The boundary condition is

$$u(0, t) = \sin(\bar{\omega}t), \qquad u_x(l, t) = 0 \qquad (25)$$
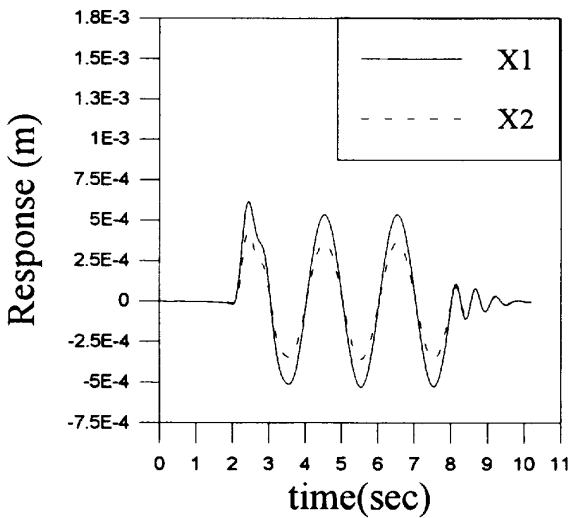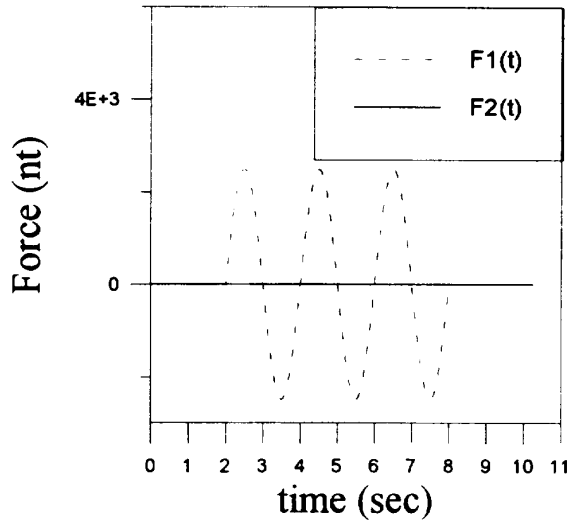
where $\bar{\omega} = \pi$.

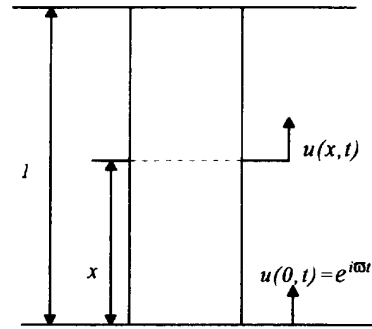**Fig. 5.** Responses of the two-DOF system.

Fig. 6. A shear beam model (continuous system).

The hysteresis loop at the depth l can be constructed as shown in Fig. 7 and is compared with the frequency domain approach.[14] The results are satisfactory and match well.

4 CONCLUDING REMARKS

The governing equation in the time domain for the SDOF system of hysteretic damping has been employed to construct the hysteretsis loop. The transient curve in the development from the origin to the steady state on the damping ellipse has been found. Also, numerical examples for different loss factors have been presented and the lack of dependence of the exciting frequency on the dissipation energy has been confirmed when the maximum responses are the same. Finally, the extensions to MDOF system and continuous system are shown to validate the present formulation.
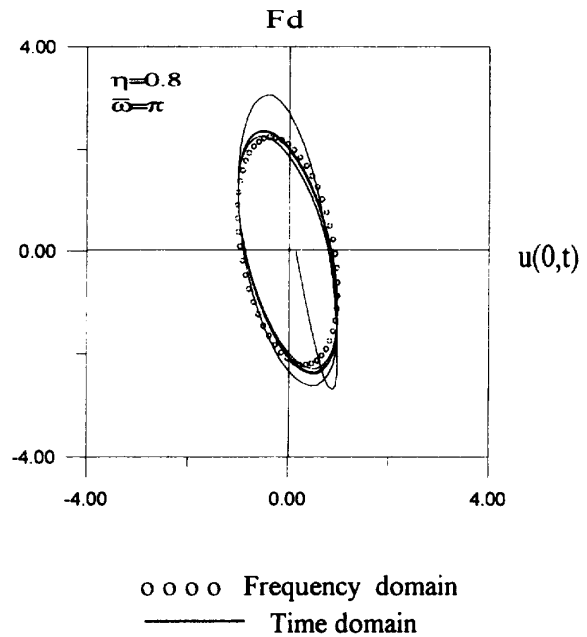

o o o o Frequency domain

—————— Time domain

Fig. 7. Hysteresis loop at $x = l$ for the shear beam.

ACKNOWLEDGEMENT

REFERENCES

1. Chen, J. T. & You, D. W., On hysteretic damping mechanism in the time domain. Proposal NSC 85-2211-E-019-004, National Taiwan Ocean University, Keelung, Taiwan, 1994 (in Chinese).
2. Chen, J. T. & You, D. W., Time-domain approach for hysteretic damping model. *Proc. 3rd National Congress Structural Engineering*, 1996 (in Chinese).
3. Chen, J. T. & You, D. W., An integral–differential equation approach for the free vibration of a SDOF system with hysteretic damping. *BETECH96 Conf.*, Hawaii, 1996.
4. Chen, L. Y., Chen, J. T., Chen, C. H. & Hong, H.-K., Free vibrations of a SDOF system with hysteretic damping. *Mech. Res. Commun.*, 1994, **21**, 599–604.
5. Crandall, S. H., A new hysteretic damping model? *Mech. Res. Commun.*, 1995, **22**, 201–202.
6. Inaudi, J. A. & Kelly, J. M., Linear hysteretic damping and the Hilbert transform. *J. Eng. Mech. ASCE*, 1995, **121**, 662–632.
7. Nashif, A. D., Jones, D. I. G. & Henderson, J. P., *Vibration Damping*. Wiley, New York, 1984.
8. Sun, C. T. & Lu, Y. P., *Vibration Damping of Structural Elements*. Prentice-Hall, Englewood Cliffs, NJ, 1995.
9. Meirovitch, L., *Elements of Vibration Analysis*. McGraw-Hill, New York, 1986.
10. Gaul, L., Bohlen, S. & Kempfle, S., Transient and forced oscillation of systems with constant hysteretic damping. *Mech. Res. Commun.*, 1985, **12**, 187–201.
11. Gaul, L., Structural damping in frequency and time domain. *Proc. 7th Int. Modal Analysis Conf.*, 1989, pp. 177–184.
12. Chen, J. T., Hong, H.-K. & Yeh, C. S., Modal reaction method for modal participation factor of support motion problems. *Commun. Numer. Meth. Engng.*, 1995, **11**, 479–490.
13. Chen, J. T., Hong, H.-K., Yeh, C. S. & Chyuan, S. W., Integral representations and regularizations for divergent series solution of a beam subjected to support motions. *Earthquake Engineering and Structural Dynamics*, 1996, **25**, 909–925.
14. Newland, D. E., *Mechanical Vibration Analysis and Computation*. Longman, London, 1989.
15. Chen, J. T., Chyuan, S. W., You, D. W. & Wong, F. C., Normalized quasi-static mass — a new definition for multi suppot motion problems. *Finite Elements in Analysis and Design*, 1996, to appear.