Analytical integrations in 3D BEM for elliptic problems: Evaluation and implementation

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SUMMARY

The present publication deals with 3D elliptic boundary value problems (potential, Stokes, elasticity) in the framework of linear, isotropic, and homogeneous materials. Numerical approximation of the unique solution is achieved by 3D boundary element methods (BEMs). Adopting polynomial test and shape functions of arbitrary degree on flat triangular discretizations, the closed form of integrals that are involved in the 3D BEMs is proposed and discussed. Analyses are performed for all operators (single layer, double layer, hypersingular). The Lebesgue integrals are solved working in a local coordinate system. For singular integrals, both a limit to the boundary as well as the finite part of Hadamard (Lectures on Cauchy’s Problem in Linear Partial Differential Equations. Yale University Press: New Haven, CT, U.S.A., 1923) approach have been considered. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: integral equations; boundary elements; analytical integrations; finite part of Hadamard

1. INTRODUCTION

Modeling elliptic problems by means of boundary integral equations (BIEs) and approximating their solution via boundary element methods (BEMs) is firmly established in the academic community as well as in industry. Several stimulating modern applications and research topics can be effectively described via BIEs: to cite but a few, size and location of tumors from temperature measurements [1], mechanics of highly non-linear material behaviors eventually with large strains and rotations [2] as well as strain gradient constitutive laws [3], mechanics of carbon nanotubes composites [4] and dislocations [5].

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In the traction equation (5), the boundary limit must be taken at a point \( \vec{p} \).

Table I. Kernels and their singularities. Here \( r = ||\vec{r}|| \). CPV stands for Cauchy’s Principal Value, whereas HFP for Hadamard’s Finite Part.

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The present note aims at providing a closed form for some ‘integrals’ involved into the BEMs approximation of 3D elliptic problems [6]. Reference is made to linear elasticity for isotropic and homogeneous materials, as a prototype of such a class of problems. Bulk forces in domain \( \Omega \) are denoted with \( \vec{f}(\vec{y}) \); the boundary of \( \Omega \) is termed \( \Gamma \) and is assumed to be Lipschitz and piecewise continuous; displacements \( \vec{u}(\vec{y}) \) are given on boundary \( \Gamma_u \subseteq \Gamma \), whereas tractions \( \vec{p}(\vec{y}) \) are given on boundary \( \Gamma_p \subseteq \Gamma \); boundary is taken such that \( \Gamma_u \cap \Gamma_p = \Gamma \) and \( \Gamma_u \cap \Gamma_p = \emptyset \); vector \( \vec{r} \) is defined by \( \vec{r} = \vec{x} - \vec{y} \); vector \( \vec{d} \) denotes its opposite: \( \vec{d} = -\vec{r} = \vec{y} - \vec{x} \) (componentwise, \( d_l = y_l - x_l \) with \( l = 1, 2, 3 \)).

The boundary integral formulation of Lamé equations [7–11] stems from Somigliana’s identity:

\[
\vec{u}(\vec{x}) + \int_{\Gamma} G_{up}(\vec{r}; \vec{l}(\vec{y})) \vec{u}(\vec{y}) \, d\Gamma_y = \int_{\Gamma} G_{uu}(\vec{r}) \vec{p}(\vec{y}) \, d\Gamma_y + \int_{\Omega} G_{uu}(\vec{r}) \vec{f}(\vec{y}) \, d\Omega_y, \quad \vec{x} \in \Omega
\] (1)

Equation (1) is the boundary integral representation (BIR) of displacements in the interior of \( \Omega \).

The traction operator can be applied to Somigliana’s identity in the interior of \( \Omega \), thus obtaining the BIR of tractions on a surface of normal \( \vec{n}(\vec{x}) \), \( \vec{x} \in \Omega \):

\[
\vec{p}(\vec{x}, \vec{n}(\vec{x})) + \int_{\Gamma} G_{pp}(\vec{r}; \vec{n}(\vec{x}); \vec{l}(\vec{y})) \vec{u}(\vec{y}) \, d\Gamma_y = \int_{\Gamma} G_{pu}(\vec{r}; \vec{n}(\vec{x})) \vec{p}(\vec{y}) \, d\Gamma_y + \int_{\Omega} G_{pu}(\vec{r}; \vec{n}(\vec{x})) \vec{f}(\vec{y}) \, d\Omega_y
\] (2)

Green’s functions \( G_{uu}, G_{up}, G_{pu} \) and \( G_{pp} \) may contain terms of the following kind (see Table I and the expressions of kernels in Appendix A):

\[
\frac{d_1^\alpha d_2^\beta d_3^\gamma}{r^{2+\beta+\gamma+s}}, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad s = 1, 2, 3
\] (3)

BIEs for the linear elastic problem can be derived from BIRs (1) and (2) by performing the boundary limit \( \Omega \ni \vec{x} \to \vec{x} \in \Gamma \). The limit process triggers off singularities of Green’s functions,

\[\text{Equation (1) is based on Green’s functions } G_{uu} \text{ and } G_{up}. \text{ Unfortunately, the notation internationally used for Green’s functions is not unambiguous. In this note, it follows from [9]. Accordingly, } G_{uu} \text{ stands for the Kelvin tensor, which represents components } u_l \text{ of the displacement vector } \vec{u} \text{ at a point } \vec{x} \text{ due to a unit force concentrated in space (point } \vec{y} \text{) and acting on the unbounded elastic space } \Omega_\infty \text{ in the direction } \vec{f}. \text{ Kernel } G_{ap} \text{ is derived from the Kelvin tensor by means of the boundary traction operator. One finds in [9, 12], among others, a detailed description of the physical meaning of all Green’s functions involved in this note.}
\]

\[\text{In the traction equation (5), the boundary limit must be taken at a point } \vec{x} \in \Gamma \text{ with a well-defined normal vector } \vec{n}(\vec{x}). \text{ Strong and hypersingular kernels generate free-terms—with the notation of [13] they will be termed as } \frac{1}{2} \vec{f} \text{ and } \frac{1}{2} \vec{f} \text{—in the limit process such that } \frac{1}{2} \vec{f}(\vec{x}) = \xi(\vec{x}) = \frac{1}{2} \vec{f} \text{ for smooth boundaries [13–19], whereas special care is required for the discrete problem (see again [13]).}
\]
due to terms (3). They have been extensively investigated. Kernel $G_{uu}$, which appears in the Single Layer Potential operator $V: H^{-1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$, shows an integrable singularity (named ‘weak’); kernels $G_{up}$, within the Double Layer Potential operator $K: H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)$, and $G_{pu}$, within the Adjoint Double Layer Potential operator $K': H^{-1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$, present a strong singularity $O(r^{-2})$; kernel $G_{pp}$, which pertains into the Hypersingular Integral Operator $D: H^{1/2}(\Gamma) \rightarrow H^{-1/2}(\Gamma)$, shows a singularity $O(r^{-3})$ greater than the dimension of the integral. Assuming boundary $\Gamma$ to be smooth, the following BIEs come out:

\[
\frac{1}{2} \mathbf{u}(\mathbf{x}) + \int_{\Gamma_p} G_{up}(\mathbf{r}; \mathbf{l}(\mathbf{y})) \mathbf{u}(\mathbf{y}) \, d\Gamma_y + \int_{\Gamma_u} G_{up}(\mathbf{r}; \mathbf{l}(\mathbf{y})) \mathbf{\tilde{u}}(\mathbf{y}) \, d\Gamma_y = \int_{\Gamma_u} G_{uu}(\mathbf{r}) \mathbf{p}(\mathbf{y}) \, d\Gamma_y + \int_{\Gamma_p} G_{uu}(\mathbf{r}) \mathbf{\tilde{p}}(\mathbf{y}) \, d\Gamma_y + \int_{\Omega} G_{uu}(\mathbf{r}) \mathbf{f}(\mathbf{y}) \, d\Gamma_y, \quad \mathbf{x} \in \Gamma
\]

(4)

\[
\frac{1}{2} \mathbf{p}(\mathbf{x}) + \int_{\Gamma_p} G_{pp}(\mathbf{r}; \mathbf{n}(\mathbf{y}); \mathbf{l}(\mathbf{y})) \mathbf{u}(\mathbf{y}) \, d\Gamma_y + \int_{\Gamma_u} G_{pp}(\mathbf{r}; \mathbf{n}(\mathbf{y}); \mathbf{l}(\mathbf{y})) \mathbf{\tilde{u}}(\mathbf{y}) \, d\Gamma_y = \int_{\Gamma_u} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{p}(\mathbf{y}) \, d\Gamma_y + \int_{\Gamma_p} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{\tilde{p}}(\mathbf{y}) \, d\Gamma_y + \int_{\Omega} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{f}(\mathbf{y}) \, d\Gamma_y, \quad \mathbf{x} \in \Gamma
\]

(5)

Equation (4) is referred to as ‘displacement equation’, whereas Equation (5) is named as ‘traction equation’: they permit to derive the Calderon Projector for the elastostatic operator. After imposing the fulfillment of Equation (4) on the Dirichlet boundary $\Gamma_u$ and of Equation (5) on the Neumann boundary $\Gamma_p$, the following linear boundary integral problem comes out:

\[
\begin{bmatrix}
V[.]  & -K[.] \\
-K'[.]  & -D[.]
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}(\mathbf{y}) \\
\mathbf{u}(\mathbf{y})
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{f}'(\mathbf{x}) \\
\mathbf{f}^p(\mathbf{x})
\end{bmatrix}, \quad \mathbf{x} \in \Gamma_u
\]

(6)

Integral operators $V, K, K', D$ are defined by comparison in Equations (4), (5), and (6). Vectors $\mathbf{f}'(\mathbf{x}), i = u, p$, that gather all data (i.e. $\mathbf{\tilde{p}}, \mathbf{\tilde{u}}, \mathbf{f}$) follow:

\[
\mathbf{f}^u(\mathbf{x}) = \frac{1}{2} \mathbf{\tilde{u}}(\mathbf{x}) - \int_{\Gamma_p} G_{uu}(\mathbf{r}) \mathbf{\tilde{p}}(\mathbf{y}) \, d\Gamma_y + \int_{\Gamma_u} G_{up}(\mathbf{r}; \mathbf{l}(\mathbf{y})) \mathbf{\tilde{u}}(\mathbf{y}) \, d\Gamma_y - \int_{\Omega} G_{uu}(\mathbf{r}) \mathbf{f}(\mathbf{y}) \, d\Omega_y, \quad \mathbf{x} \in \Gamma_u
\]

\[
\mathbf{f}^p(\mathbf{x}) = -\frac{1}{2} \mathbf{\tilde{p}}(\mathbf{x}) + \int_{\Gamma_p} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{\tilde{p}}(\mathbf{y}) \, d\Omega_y - \int_{\Gamma_u} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{\tilde{u}}(\mathbf{y}) \, d\Gamma_y + \int_{\Omega} G_{pu}(\mathbf{r}; \mathbf{n}(\mathbf{y})) \mathbf{f}(\mathbf{y}) \, d\Omega_y, \quad \mathbf{x} \in \Gamma_p
\]

Integral problem (6) can be given the compact form

\[
L[\mathbf{z}] = \mathbf{f}
\]

(7)
with all terms defined by comparison. Unknown vector \( \mathbf{z} \) is made of tractions (Neumann data) \( \mathbf{p} \) on the Dirichlet boundary \( \Gamma_u \) and displacements (Dirichlet data) \( \mathbf{u} \) on the Neumann boundary \( \Gamma_p \). Denote with \( Z_L \) the domain of \( L \) and define bilinear form \( \mathcal{A}_L : Z_L \times Z_L \to R \) as

\[
\mathcal{A}_L (\mathbf{a}, \mathbf{b}) \equiv \int_{\Gamma_u} (V[\mathbf{p}^a](\mathbf{x}) - K[\mathbf{u}^a](\mathbf{x}))\mathbf{p}^b(\mathbf{x}) \, d\Gamma_x + \int_{\Gamma_p} (-K'[\mathbf{p}^a](\mathbf{x}) - D[\mathbf{u}^a](\mathbf{x}))\mathbf{u}^b(\mathbf{x}) \, d\Gamma_x \quad (8)
\]

It can be proved—starting from the property of reciprocity [7]—that bilinear form \( \mathcal{A}_L \) is symmetric:

\[
\mathcal{A}_L (\mathbf{a}, \mathbf{b}) = \mathcal{A}_L (\mathbf{b}, \mathbf{a}) \quad \forall \mathbf{a}, \mathbf{b} \in Z_L
\]

As a consequence of the mapping properties of operators \( V \) and \( D \), problem (6) is uniquely solvable provided that some conditions are fulfilled [20]. The solution is shown to be a critical point of functional

\[
\Psi[\mathbf{z}] = \frac{1}{2} \mathcal{A}_L (\mathbf{z}, \mathbf{z}) - \int_{\Gamma} \mathbf{z}(\mathbf{x})f(\mathbf{x}) \, d\Gamma_x 
\]

Let \( h > 0 \) be a parameter and let \( [\mathbf{p}_h(\mathbf{y}), \mathbf{u}_h(\mathbf{y})]^T \) defined as \( \mathbf{z}_h \in Z_{L,h} \) be a (discrete) approximation of the unknown vector field \( \mathbf{z} \), denoting with \( Z_{L,h} \) a family of finite-dimensional subspaces of \( Z_L \) such that

\[
\forall \mathbf{z} \in Z_L, \quad \inf_{\mathbf{z}_h \in Z_{L,h}} \| \mathbf{z} - \mathbf{z}_h \| \to 0 \quad \text{as} \quad h \to 0 
\]

(10)

Discretization (10) allows to transform integral problem (7) into a set of algebraic equations. Two main techniques have been successfully developed to this aim: the collocation (CBEM) [11] and the symmetric Galerkin [9] method (SGBEM).

The CBEM\(^1\) requires the fulfillment of integral equation (4) onto a selected set of collocation points \( \mathbf{x}_i \in \Gamma \). The evaluation of ‘integrals’

\[
\int_{\Gamma_x} G_{rs}(\mathbf{x}_i - \mathbf{y})\phi_h(\mathbf{y}) \, d\Gamma_y, \quad r, s \in \{u, p\}
\]

(11)

is required, where symbol \( \phi_h(\mathbf{y}) \) denotes the \( n \)th scalar shape function. The SGBEM approximation of (7) consists of finding \( \mathbf{z}_h \in Z_{L,h} \), critical point of the function:

\[
\Psi[\mathbf{z}_h] = \frac{1}{2} \mathcal{A}_L (\mathbf{z}_h, \mathbf{z}_h) - \int_{\Gamma} \mathbf{z}_h(\mathbf{x})f(\mathbf{x}) \, d\Gamma_x
\]

By imposing the stationarity of \( \Psi[\mathbf{z}_h] \) respect to the set of nodal values, one deals with the integrals of the following form:

\[
\int_{\Gamma_x} \phi_k(\mathbf{x}) \int_{\Gamma_x} G_{rs}(\mathbf{x}, \mathbf{y})\phi_h(\mathbf{y}) \, d\Gamma_y \, d\Gamma_x, \quad r, s \in \{u, p\}
\]

(12)

where \( \phi_k(\mathbf{x}), \phi_h(\mathbf{y}) \) are scalar test and shape functions.

\(^1\)In the modeling of fracture mechanics problems, an insurmountable mathematical difficulty arises in applying the CBEM making use of the displacement equation only (see e.g. [21, 22]). Several special techniques have been devised to overcome this mathematical degeneracy: among others, the special Green’s functions methods [23], the zone method [24] and the Dual BEM [25].

Denoting with $\Gamma_h$ a flat triangulation of boundary $\Gamma$ and with $T_j \subset \Gamma_h$ its generic triangle, the paper is devoted to the analytical integration of

$$\int_{T_j} G_{rs}(x-y)\phi_n(y)\,d\Gamma_y, \quad r, s \in \{u, p\}$$

(13)

when shape functions $\phi_n(y)$ are polynomials of arbitrary degree. When point $x \in T_j$, integral (13) may not exist in a classical sense. In such cases, the continuity (with respect to $x$) between Hadamard’s finite part (HFP) of divergent integral (13) and the Lebesgue integral is shown. To achieve this aim, the HFP has been directly evaluated as first; further, the limit process to the boundary $\Omega \ni x \to x \in \Gamma$ has been performed—see Section 3.

The implementation requires particular care when point $x$ belongs to the lines that define the sides of $T_j$, even if $x \notin \partial T_j$. Section 4.3 deals with the implementation details. Computational effort of analytical integrations is discussed in Section 4.4. Finally, two benchmarks are considered in Section 4.5, in order to check the analytical integrations and the implementation: they conclude the paper.

2. PROBLEM FORMULATION AND MAJOR RESULTS

2.1. Approximation of unknown fields

Let $\Gamma_h$ be a triangulation of boundary $\Gamma$, $T_j \in \Gamma_h$ its generic triangle and $a_n$ its generic node. Collect in set $\mathcal{T}_n$ all triangles $T_j$ of $\Gamma_h$ sharing node $a_n$ (see an example in Figure 1). Assume over $T_j$ a local (lagrangian) basis $\varphi_j := \{\phi_j^1, \phi_j^2, \ldots, \phi_j^M(j)\}$ and denote with $\phi_j^{n(j)}$ the unique element of $\varphi_j$ such that $\phi_j^{n(j)}(a_n) = 1$. Piecewise continuous shape functions $\phi_n(x)$ (see Figure 1) are defined as follows:

$$\phi_n \in C^0(\Gamma_h), \quad \text{supp}(\phi_n) = \mathcal{T}_n, \quad \phi_n|_{T_j} = \phi_j^{n(j)}$$

(14)

Discontinuous shape functions $\phi_n(x)$ are taken as elements of local (lagrangian) basis $\varphi_j$.

Consider a direction $l$. For scalar problems, $l = 1$ will be generally omitted; for vector problems, $l = 1, 2, 3$. Collect in vector $\phi_l^u$ all piecewise continuous shape functions defined by Equation (14) for the discrete approximation $u_h(y)$ of the Dirichlet field $u(y)$ relevant to direction $l$

$$\phi_l^u(y) = \{\phi_n(y) \mid n = 1, 2, \ldots, N_l\}$$

(15)

Figure 1. A possible picture of triangle $T_j$, set $\mathcal{T}_n$, local $\phi_j^{n(j)}(x)$ and global $\phi_n(x)$ shape functions.
Figure 2. An effective representation for \( \varphi_n^{(j)} \) in local coordinate system \( \mathcal{L} \). Here \( H_j = \sqrt{1 + a^2} H_j \).

Equation \( y_3 - a y_2 = 0 \) defines the slanting side of triangle \( T_j \), as here depicted. The second slanting side is defined by \( y_3 - b y_2 = 0 \), with \( b > a \). Finally, the third side is defined by \( y_2 = \tilde{y}_2 \).

\( N_l \) is the number of nodal unknowns in the direction \( l \). Vector \( \phi_l^p \) for the discrete approximation \( p_h(y) \) of Neumann field \( p(y) \) relevant to direction \( l \) is defined analogously; vector \( \phi_l^p \) may contain discontinuous shape functions, depending on the geometry.

Unknown vector fields can be written, therefore, as

\[
\mathbf{u}_h(y) = \sum_l e_l \otimes \phi_l^u(y) \hat{u}_l, \quad \mathbf{p}_h(y) = \sum_l e_l \otimes \phi_l^p(y) \hat{p}_l
\]  

In the former equation: (i) tensor product \( \otimes : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n \times m} \) is defined as: \( (a \otimes b) c = (b \cdot c) a \); (ii) for vector problems, \( e_l \) is the unit vector in direction \( l \); for scalar problems, it is merely the number 1; (iii) \( \hat{u}_l, \hat{p}_l \) are nodal values for displacements and tractions in direction \( l \).

2.2. Local basis

A local coordinate system \( \mathcal{L} = \{ y_1, y_2, y_3 \} \) is defined over triangle \( T_j \) in this way: (i) a vertex of \( T_j \) is the origin; (ii) plane \( y_1 = 0 \) contains \( T_j \); (iii) plane \( y_3 = 0 \) is orthogonal to the side of \( T_j \) that does not include the origin. \( T_j \) is defined in \( \mathcal{L} \) as

\[
T_j := \{ y \in \mathbb{R}^3 \text{ s.t. } y_1 = 0; \ 0 < y_2 < \tilde{y}_2; \ ay_2 - y_3 < 0; \ by_2 - y_3 > 0; \ a < b \}
\]

Coefficients \( a \) and \( b \) denote the slope of the two sides of \( T_j \) that include the origin (see Figure 2).

Select arbitrarily one of these two sides, say \( y_3 - a y_2 = 0 \). Denote with \( H_j \) the height of \( T_j \)—see Figure 2. Shape functions \( \varphi_n^{(j)}(y) \) in (15) can be expressed in local reference \( \mathcal{L} \) as

\[
\varphi_n^{(j)}(y) = y_3^T \Lambda_j^n y_2
\]  

Vectors \( y_3 \) and \( y_2 \) are defined by

\[
y_3^T = \{1, y_3, y_3^2, \ldots\}, \quad y_2^T = \{1, y_2, y_2^2, \ldots\}
\]
and matrix $\Lambda_j^n$ depends on node $a_n$. For instance, in the six-node element of Figure 2 with reference to node $\tilde{a}_n$, one writes matrix $\Lambda_j^n$ as

$$
\Lambda_j^n = \begin{bmatrix}
0 & \frac{4a}{(a-b)y_2} & -\frac{4a}{(a-b)y_2^2} \\
\frac{4}{(a-b)y_2} & \frac{4}{(a-b)y_2^2} & 0
\end{bmatrix}
$$

In order to solve ‘integral’ (13), a variable change is worth. For the generic component of vectors $y_2^T, y_3^T$ in Equation (17), the binomial expansion reads as

$$
y_{2}^{i} = (x_{2} + d_{2})^{i} = \sum_{k=0}^{i} \binom{i}{k} x_{2}^{(i-k)} d_{2}^{k}, \quad \alpha = 2, 3
$$

with vector $d = -r = y - x$ (componentwise, $d_2 = y_2 - x_2$ with $\alpha = 2, 3$) already defined in the introduction. Shape functions $\phi_j^n(y)$ in (17) can be written as follows:

$$
\phi_j^n(y) = d_2^T X_3^{(3)} \Lambda_j^n X_2^{(2)} d_2
$$

where

$$
d_2^T = \{1, d_2, d_2^2, \ldots\}, \quad X_{ij}^{(a)} = \begin{cases} i-j \binom{i}{j-1} x_2^{(i-j)}, & i, j = 1, 2, \ldots; \quad \alpha = 2, 3 \end{cases}
$$

For instance, if one considers linear shape functions, the one referred to the node at the origin reads as

$$
\phi_j^{n(j)}(y) = X_3^{(3)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\
0 \end{bmatrix} d_2
$$

with

$$
X_2^{(2)} = \begin{bmatrix} 1 & 0 \\
x_2 & 1 \end{bmatrix}, \quad X_3^{(3)} = [1]
$$

2.3. Problem formulation

Integral (13) to be solved comes out by substituting the unknown Dirichlet $u$ and Neumann $p$ fields with the discrete approximations (16) into BIEs (4)–(5). By definition (14), ‘integral’ (13) can be casted in the form as

$$
\int_{\text{supp}(\phi_n)} G_{rs}(x, y) \phi_n(y) \, d\Gamma_y = \sum_j \int_{T_j} G_{rs}(x, y) \phi_j^{n(j)}(y) \, d\Gamma_y \text{ def } \sum_j F_{rsj}^n(x)
$$

Because of the simplicity of form (19), it may be computationally worth for linear shape functions to consider three different local reference systems—one at each vertex—instead of making use of expression (17), what would lead to a more involved matrix $\Lambda_j^n$. 

and further simplified in local reference $L'$ as

$$F_{rs}^n_j(x) = \int_{-x}^{y_2 - x_2} \int_{bd_2 + k_b}^{bd_2 + k_b} G_{rs}(d) d_3 d_2 d_3 d_2 \left( \begin{array}{c} 1 - \frac{x_2}{y_2} \\ \frac{1}{y_2} \\ \frac{1}{y_2} \end{array} \right)$$

(21)

Coefficient $k_a$ and $k_b$ are defined as $k_a = ax_2 - x_3$ and $k_b = bx_2 - x_3$. The identities $k_a = 0$ and $k_b = 0$ materialize the two sides of $T_j$ that include the origin (see Figure 2).

In the case of linear shape functions, vector $d_2 = \{1, d_2\}$ and integral (21) takes the following easy form:

$$F_{rs}^n_j(x) = \int_{-x_3}^{y_2 - x_2} \int_{bd_2 + k_b}^{bd_2 + k_b} G_{rs}(d) d_3 d_3 d_2 d_3 \left( \begin{array}{c} 1 - \frac{x_2}{y_2} \\ \frac{1}{y_2} \end{array} \right)$$

(22)

For scalar problems $F_{rs}^n_j(x)$ is a scalar function, whereas $\kappa_{rs}$ is a vector of dimension 2. For vector problems, $F_{rs}^n_j(x)$ is a matrix of the same order of kernel $G_{rs}$, whereas $\kappa_{rs}$ is a third order matrix, whose third dimension is equal to 2.

2.4. Major result of the paper

In what follows, analytical integrations will be carried out with reference to integral (21), but tables will be presented only for $\kappa_{rs}(x)$ for paucity of space. In such a case, it turns out that in the local coordinate system $L'$

$$\kappa_{rs}(x) = \kappa \kappa_{rs}(x, d_2, d_3) \left| \begin{array}{c} d_3 = bd_2 + k_b \\ d_2 = y_2 - x_2 \\ d_3 = ad_2 + k_a \\ d_2 = -x_2 \end{array} \right.$$ (23)

$\kappa \in \mathbb{R}$ depends on material parameters of the problem under consideration. $\kappa_{rs}$ is defined by a sum of basic terms as follows:

$$\kappa_{rs}(x, d_2, d_3) = L^r_{rs} \log(\zeta + r) + A^r_{rs} \arctanh \left( \frac{d_3}{r} \right) + \mathbb{R}^r_{rs} \mathbb{R}^r_{rs} + \mathbb{S}^r_{rs} + \mathbb{H}^r_{rs}$$ (24)

where

- $L^r_{rs}(x, d_2, d_3)$ is the Lebesgue integral of $r^{-3}$ over $T_j$. It has been discussed in great detail in [26] and will be shortly summarized in Section 4.2.
- $L^r_{rs}, A^r_{rs}, \mathbb{R}^r_{rs}, \mathbb{S}^r_{rs}, \mathbb{H}^r_{rs}$ are vectors (for scalar problems) or third-order tensors (for vector problems). Their expressions are collected in Appendix C for potential problems. Owing to paucity of space, expressions for linear elasticity have not been included here and will appear in a separate report.

$\kappa_{rs}(x)$ depends on the kernel $G_{rs}$, on the selected element $T_j$, and on the position of point $x$. Weakly singular kernel $G_{uu}(d)$, strongly singular kernels $G_{up}(d, l(y))$ and $G_{pu}(d, n(x))$, and hyper-singular kernel $G_{pp}(d, n(x), l(y))$ are singular with respect to $y$ depending on the position of $x$ with respect to $T_j$. The item $x \in T_j$ (that, for all kernels, leads to a Lebesgue inner integral) and the item $x \not\in T_j$ (that leads to an improper integral for $G_{uu}$, to a Cauchy’s Principal Value (from now on shortened as CPV) for $G_{up}$ and $G_{pu}$, and to a HFP for $G_{pp}$) will be therefore separately dealt with.
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Formulae (23, 24) will be derived in the next section and particularized to the single layer potential (Equations (43, 44)), to the double layer potential and its adjoint (Equations (50)–(52)), and to the hypersingular integral operator (Equations (60), (61)). This derivation requires a certain amount of details. The reader who does not feel interested in them may jump to Section 4 where formulae (23, 24) will be discussed.

3. ANALYTICAL INTEGRATIONS

Green’s functions $G_{uu}$, $G_{up}$, $G_{pu}$ and $G_{pp}$ contain terms (3). Once they are placed into $F_{r}\eta_j(x)$, the following basic integrals must be dealt with:

$$\int_{a_{d_2}+k_a}^{b_{d_2}+k_b} \frac{d^k_{3}}{r^{2m+1}} \, dd_3, \quad k, m \in \mathbb{N}_0$$

Identity

$$\frac{x^{2k}}{x^2+x^2} = (-1)^k \frac{x^{2k}}{x^2+x^2} + \sum_{j=1}^{k} \binom{k}{j}(-1)^{k-j}(x^2+x^2)^{j-1}(x^2)^{k-j}$$

(25)

which comes out from the binomial expansion rule, permits to obtain the following recursive relationship that seems to be useful for analytical integrations:

$$\frac{d^k_{3}}{r^{2m+1}} = (-x^2)^{\hat{k}} \frac{d^{[2]}_{3}}{r^{2m+1}} + \sum_{j=1}^{\hat{k}} \binom{\hat{k}}{j}(-1)^{j-\hat{k}} \sum_{h=0}^{j-1} \binom{j-1}{h} \frac{d^{2h+k_{[2]}}_{3}}{r^{2m-1}}, \quad k, m \in \mathbb{N}_0$$

(26)

where $x^2 = d_1^2 + d_2^2$ is the squared projection of the distance on the plane $d_3 = 0$. Here and in the remainder of the section, the following notation will be considered:

$k = k + 2$ division in $\mathbb{N}$.

$k_{[2]} = k - 2\hat{k}$ remainder of $k + 2$ in $\mathbb{N}$.

3.1. Lebesgue integrals

Consider $x \not\in T_j$, which implies $r \neq 0$ and the Lebesgue nature of integral (21). Exploiting recursively identity (26), the first contribution of $F_{r}\eta_j(x)$, namely

$$\int_{a_{d_2}+k_a}^{b_{d_2}+k_b} G_{rj}(d) d^3_{3} \, dd_3$$

is reduced to the sum of a set of basic integrals; the following identities, which can be easily proved by induction, are required for most kernels:

$$\int \frac{x^j}{\sqrt{2x^2+x^2}} \, dx = \mu(j)x^l \log(x+\sqrt{2x^2+x^2}) + \sum_{k=0}^{j-1} \eta(k,j)x^{2(j-k-1)}x^{2k+1-j+[2]}\sqrt{2x^2+x^2}$$

(27)

$$\int \frac{x^{n_{[2]}}}{(x^2+x^2)^{1/2}} \, dx = \frac{x-(x^2+x)(n_{[2]})}{x^2\sqrt{2x^2+x^2}}$$

(28)

\[
\frac{x^{n(2)}}{(x^2 + x^2)^2 \sqrt{x^2 + x^2}} \, dx = \frac{3 x^2 x + 2 x^3 - (x^4 + 3 x^2 x + 2 x^3) n_2}{3 x^4 (x^2 + x^2)^{3/2}}
\] (29)

\[
\frac{x^{n(2)}}{(x^2 + x^2)^3 \sqrt{x^2 + x^2}} \, dx = \frac{15 x^3 x + 20 x^2 x^3 + 8 x^5 (1 - n_2) - 3 x^6 n_2}{15 x^6 (x^2 + x^2)^{5/2}}
\] (30)

where
\[\hat{j} = j + 2\] division in \(\mathbb{N}\).
\[\hat{j}_{2} = j - 2\] remainder of \(j + 2\) in \(\mathbb{N}\).
\[\tilde{j} = j - \hat{j}\] complementary part of \(j + 2\) in \(\mathbb{N}\).
\[\mu(j) = (1 - j_{2})(-1)^{\hat{j} \frac{j_{2}}{(j_{2}+1)} j_{2} \left(1 - j_{2}(j_{2}+1)\right)}^{2} + j_{2}(2k) \left(\frac{j_{2}}{k^{2}-j_{2}}\right)^{2}\]

In view of (27)–(30), it follows:
\[
\int_{ad_{2}+ka}^{bd_{2}+kb} G_{rs}(d) d_{3}^{2} \, dd_{3} = \sum_{l,m=0}^{N(k,l,m)} \frac{A_{lm}(x)}{d_{1}^{2} + d_{2}^{2}} \frac{d_{3}^{2}}{r^{2m-1}} + \sum_{j=0}^{k} B^{j}(x) d_{2}^{2} \log(d_{3} + r) \bigg|_{d_{3}=ad_{2}+ka}^{d_{3}=bd_{2}+kb}
\] (31)

For scalar problems \(A_{lm}^{j}\) and \(B^{j}\) are scalar functions, whereas for vector problems they are matrices’ functions of the same order (here termed \(N_{rs}:\) for elasticity, \(N_{rs} = 3\) of kernel \(G_{rs}\). Of course, \(A_{lm}^{j}\)
and \(B^{j}\) depend upon the considered kernel. Integral (31) can be recast in the vector formalism of Equation (21), namely
\[
\int_{ad_{2}+ka}^{bd_{2}+kb} G_{rs}(d) d_{3}^{T} \, dd_{3} = \sum_{l,m=0}^{3} \frac{\tilde{d}^{T}_{2} A_{lm}(x)}{d_{1}^{2} + d_{2}^{2}} + \tilde{d}_{2}^{T} B(x) \log(d_{3} + r) \bigg|_{d_{3}=ad_{2}+ka}^{d_{3}=bd_{2}+kb}
\] (32)

Vectors \(\tilde{d}_{2}\) and \(\tilde{d}_{2}\) are special instances of \(d_{2}\), with different cardinality, here termed as \(N_{2}\) and \(N_{2}\), respectively; the latter is equal to the cardinality of \(d_{3}\), here termed as \(N_{3}\); \(N_{2}\) depends also upon \(l\) and \(m\). For scalar problems \(A_{lm}^{j}\) is a matrix \(N_{2} \times N_{3}\) of functions, whereas for vector problems \(A_{lm}^{j}\) is a fourth-order matrix \(N_{2} \times (N_{rs} \times N_{rs}) \times N_{3}\); analogously, for scalar problems \(B\) is a matrix \(N_{2} \times N_{3}\) of functions, whereas for vector problems \(B\) is a fourth-order matrix \(N_{2} \times (N_{rs} \times N_{rs}) \times N_{3}\).

Substituting expression (32) into \(F_{j}^{n}(x)\), it becomes
\[
F_{j}^{n}(x) = \sum_{l,m=0}^{3} \int_{-y_{2}}^{y_{2}} \tilde{d}_{2}^{T} A_{lm}^{n}(x) X^{(2)} X^{(3)} d_{2} \bigg|_{d_{3}=ad_{2}+ka}^{d_{3}=bd_{2}+kb} dd_{2}
\]
\[
+ \int_{-y_{2}}^{y_{2}} \hat{d}_{2}^{T} B X^{(3)} A_{j}^{n} X^{(2)} d_{2} \log(d_{3} + r) \bigg|_{d_{3}=ad_{2}+ka}^{d_{3}=bd_{2}+kb} dd_{2}
\]
with \( C_{lm}^n \) and \( D_j^n \) defined by comparison. To completely solve (21), only integrals

\[
\int_{-x_2}^{x_2} \frac{d^b_j}{(d^2_1 + d^2_2)^{r/2m-1}} \left| \begin{array}{c} d^3 = bd_j + k_b \\ d_3 = ad_2 + k_a \end{array} \right. \, dd_2, \quad l, m = 0, 1, 2, 3, \quad h \in \mathbb{N}_0
\]  

(34)

\[
\int_{-x_2}^{x_2} d^b_j \log(d_3 + r) \left| \begin{array}{c} d^3 = bd_j + k_b \\ d_3 = ad_2 + k_a \end{array} \right. \, dd_2, \quad h \in \mathbb{N}_0
\]  

(35)

are required.

### 3.1.1. Single layer operator. Matrices \( C_{lm}^n \) vanish for \( l, m = 2, 3 \). Only the following integrals are of interest for the single layer potential:

\[
\int_{-x_2}^{x_2} \frac{d^b_j}{r} \left| \begin{array}{c} d^3 = bd_j + k_b \\ d_3 = ad_2 + k_a \end{array} \right. \, dd_2
\]  

(36)

\[
\int_{-x_2}^{x_2} \frac{d^b_j}{x_1^2 + d^2_2} \left| \begin{array}{c} d^3 = bd_j + k_b \\ d_3 = ad_2 + k_a \end{array} \right. \, dd_2
\]  

(37)

\[
\int_{-x_2}^{x_2} d^b_j \log(d_3 + r) \left| \begin{array}{c} d^3 = bd_j + k_b \\ d_3 = ad_2 + k_a \end{array} \right. \, dd_2
\]  

(38)

Making use of the affine transformation

\[
\zeta_2 = \frac{d_2 + b d_3}{\sqrt{1 + b^2}}
\]  

(39)

integral (36) becomes

\[
\sum_{j=0}^{k} \binom{k}{j} \frac{(-b k_b)^{k-j}}{(\sqrt{1 + b^2})^{2k+1-j}} \int_{b k_b/(\sqrt{1 + b^2}) + \sqrt{1 + b^2}(y_2 - x_2)}^{\sqrt{1 + b^2} x_2} \frac{\gamma^j_{\zeta_2}}{\sqrt{d_1^2 + \frac{k_b^2}{1 + b^2} + r_2^2}} \, d\zeta_2
\]  

(40)

which has a closed form owing to outcome (27). By defining with \( \gamma : \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R} \) the function

\[
\gamma(k, j, h, d) \xlongleftarrow{} \binom{k}{j} \left( \binom{j-1}{h} \right) (-1)^{k-j} (d_1^2)^{\zeta_2 - 1 - h}
\]  

(41)
and by means of identity (25), Equation (37) will be rewritten as

\[
(-1)^k \hat{d}^2_1 \int_{-x_2}^{y_2-x_2} \frac{d_2^{k[2]}}{(x_1^2 + d_2^2)^2} \left| \sum_{j=1}^{j-1} \sum_{h=0}^\Delta \gamma(k, j, h, d_1) \right| dd_2 + \frac{\hat{k}}{a} \int_{-x_2}^{y_2-x_2} \frac{d_2^{k[2]+2h}}{(x_1^2 + d_2^2) r} \left| \sum_{j=1}^{j-1} \sum_{h=0}^\Delta \gamma(k, j, h, d_1) \right| dd_2
\]

Integral (42) has a closed form in view of (40), (A2) of Appendix B. Finally, the closed form of integral (38) is given in Appendix B by Equation (A6).

Algebraic manipulations lead from (40), (42) and (A6) to the following tabular expression:

\[
\hat{\kappa}_{uu}(\mathbf{x}) = \kappa \hat{K}_{uu}(\mathbf{x}, d_2, \vartheta d_2 + k_d) | \vartheta = 0 | \left| \sum_{j=1}^{j-1} \sum_{h=0}^\Delta \gamma(k, j, h, d_1) \right| d_2 = -x_2
\]

with

\[
\hat{K}_{uu}(\mathbf{x}, d_2, d_3) = \ln^{uu} \log(\zeta_2 + r) + \ln^{uu} \arctanh \frac{d_3}{r} + \ln^{uu} I_\Delta^{-3} (\mathbf{x}, d_2, d_3) + \ln^{uu} r
\]

For potential problems, \( \kappa = (4\pi)^{-1} \) whereas for linear elasticity, \( \kappa = (16G(1-v))^{-1} \). Outcome (43) can straightforwardly be extended to polynomial shape functions of arbitrarily degree over flat triangles accordingly to (33).

### 3.1.2. Double layer operator.

Besides (36)–(38), one has to deal with the following integrals:

\[
\int_{-x_2}^{y_2-x_2} \frac{d_2^{k[2]}}{(x_1^2 + d_2^2)^2} \left| \sum_{j=1}^{j-1} \sum_{h=0}^\Delta \gamma(k, j, h, d_1) \right| dd_2
\]

By means of identities (25) and (41), integral (45) can be rewritten as

\[
(-1)^k \hat{d}^2_1 \int_{-x_2}^{y_2-x_2} \frac{d_2^{k[2]}}{(x_1^2 + d_2^2)^2} \left| \sum_{j=1}^{j-1} \sum_{h=0}^\Delta \gamma(k, j, h, d_1) \right| dd_2
\]

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Integral (48) has a closed form in view of (A3) of Appendix B. Making use of affine transformation (39), integral (46) becomes

$$\sum_{j=0}^{k} \binom{k}{j} \frac{(-b k_b)^{k-j}}{(\sqrt{1+b^2})^{2k+1-j}} \int \left(\frac{b k_b}{\sqrt{1+b^2}+\sqrt{1+b^2(y_2-x_2)}}\right)^{\frac{\epsilon_j}{\sqrt{y_2}}} \frac{1}{d^2 + \frac{k_b^2}{1+b^2} + \epsilon_j^2} \frac{d \epsilon}{d_3} \frac{d_2}{d_2}$$

which has a closed form in view of (25), (27), (28). Finally, following the same path of reasoning used for integral (37), one obtains the closed form of (47) in view of (49, A8) of Appendix B.

Algebraic manipulations lead to

$$\mathbb{K}_{up}(x) = \kappa \mathbb{K}_{up}(x, d_2, \partial d_2 + k_\partial) |_{d_2 = -x_2} \quad \mathbb{K}_{pu}(x) = \kappa \mathbb{K}_{pu}(x, d_2, \partial d_2 + k_\partial) |_{d_2 = -x_2}$$

with

$$\mathbb{K}_{up}(x, d_2, d_3) = \mathbb{L}^{lu} \log(\zeta_2 + r) + \mathbb{L}^{lu} \arctan \frac{d_3}{r} + \mathbb{L}^{lu} I^3_\Delta (x, d_2, d_3) + \mathbb{L}^{lu} r + \mathbb{S}^{lu} \frac{1}{r}$$

$$\mathbb{K}_{pu}(x, d_2, d_3) = \mathbb{L}^{pu} \log(\zeta_2 + r) + \mathbb{L}^{pu} \arctan \frac{d_3}{r} + \mathbb{L}^{pu} I^3_\Delta (x, d_2, d_3) + \mathbb{L}^{pu} r + \mathbb{S}^{pu} \frac{1}{r}$$

For potential problems, $\kappa = 1/(4\pi)$ whereas for linear elasticity, $\kappa = 1/(8\pi(1-\nu))$. Identities (50), which hold for linear shape functions in view of Equation (22), can straightforwardly be extended to polynomial shape functions of arbitrarily degree over flat triangles accordingly to (33).

3.1.3. Derivative of the double layer operator. The evaluation of integrals (36)–(38) and (45)–(47) is required. Moreover
have to be faced. The path of reasoning used for solving integral (45) applies to (53) and (54) as well. They will be rewritten as

\[
(-1)^{j} d_{1}^{j} \int_{-x_{2}}^{x_{2}} \frac{d_{2}^{k_{2}}}{(x_{1}^{2} + d_{2}^{2})^{r}} \frac{1}{d_{3}=bd_{2}+k_{b}} \ dd_{2}
\]

and

\[
(-1)^{j} d_{1}^{j} \int_{-x_{2}}^{x_{2}} \frac{d_{2}^{k_{2}}}{(x_{1}^{2} + d_{2}^{2})^{r}} \frac{1}{d_{3}=bd_{2}+k_{b}} \ dd_{2}
\]

respectively. Integrals (57)–(58) have closed forms in view of (A4), (A9) of Appendix B. By means of affine transformation (39), integral (55) becomes

\[
\sum_{j=0}^{k} \binom{k}{j} \frac{(-b_{0})^{k-j}}{(\sqrt{1+b_{0}^{2}})^{2j+1}} \int_{(b_{0})/\sqrt{1+b_{0}^{2}}-\sqrt{1+b_{0}^{2}}x_{2}}^{(b_{0})/\sqrt{1+b_{0}^{2}}+\sqrt{1+b_{0}^{2}}x_{2}} \left( \frac{1}{d_{1}^{2} + 1 + b_{r}^{2} + x_{2}^{2}} \right)^{2} \frac{d_{2}}{1 + b_{r}^{2} + x_{2}^{2}} \ dd_{2}
\]

which has a closed form in view of (25), (27)–(29). Finally, following the same path of reasoning used for equation (37), the closed form of (56) comes out in view of (58) and of (A10) of Appendix B.

Algebraic manipulations lead to

\[
\mathbb{I}_{pp}(\mathbf{x}) = \mathbb{I}_{pp}(\mathbf{x}, d_{2}, d_{3})|_{\gamma=bd_{2}=x_{2}}|_{\gamma=ad_{2}=-x_{2}}
\]

with

\[
\mathbb{I}_{pp}(\mathbf{x}, d_{2}, d_{3}) = \mathbb{I}_{pp} \log(x_{2}^{2} + r) + \mathbb{P}_{pp} \ arctanh \frac{d_{3}}{r} + \mathbb{P}_{pp} r^{-3} (\mathbf{x}, d_{2}, d_{3}) + \mathbb{P}_{pp} r + \mathbb{P}_{pp} \frac{1}{r} + \mathbb{P}_{pp} \frac{1}{r^{3}}
\]

For potential problems, \( \kappa = \pi/(4\pi) \), whereas for linear elasticity, \( \kappa = G/(4\pi(1-v)) \).

3.2. Singular integrals

3.2.1. HFP. Provided that regularity requirements are satisfied, peculiarities of Green’s functions allow to perceive hypersingular integral (21) as a HFP in the limit process \( \Omega \ni \mathbf{x} \rightarrow \mathbf{x} \in \Gamma \).
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Figure 3. Geometrical description of a square neighborhood.

[27, 8, 28–30]. The definition of the finite part can be given as follows:

**Definition 1**

Let \( \varepsilon \to I(\varepsilon) \) denote a complex-valued function that is continuous in \((0, \varepsilon_0)\) and assume that

\[
I(\varepsilon) = I_0 + I_1 \log(\varepsilon) + \sum_{j=2}^{m} I_j \varepsilon^{1-j} + o(1), \quad \varepsilon \to 0
\]

where \( I_j \in \mathbb{C} \). Then \( I_0 \) is called the finite part of \( I(\varepsilon) \). In dealing with the integrals, the finite part \( I_0 \) of a (usually) divergent integral

\[
\int_{-\infty}^{+\infty} \phi(t) \, dt
\]

is denoted by the symbol

\[
\int_{-\infty}^{+\infty} \phi(t) \, dt
\]

**Definition 2**

Assume that \( x \notin \partial T_j \) and define with

\[
T^\varepsilon_j = \{ y \in T_j : |y_2 - x_2| < \varepsilon \text{ and } |y_3 - x_3| < \varepsilon \}
\]

the domain in Figure 3. In agreement with Equation (22), define with

\[
I_\Box(x, \varepsilon) \overset{\text{def}}{=} \int_{T_j \setminus T^\varepsilon_j} G_{pp}(d; e_1; e_1) \, dd_3 \, dd_2
\]

Consider first \( x \in T_j \). Elastostatic kernel \( G_{pp} \) and potential kernel \( G_{pp}^{\text{lapl}} \) simplify as

\[
G_{pp}(d; e_1; e_1) = \frac{G_v}{(1 - v)} \left\{ 2(e_1 \otimes e_1) + \frac{(1-2v)}{v} I + 3 \frac{d \otimes d}{r^2} \right\} \frac{1}{4\pi r^3} , \quad G_{pp}^{\text{lapl}}(d; e_1; e_1) = -\frac{\chi}{4\pi r^3}
\]

in local coordinate system \( \mathcal{L} \). With reference to elastostatic kernel \( G_{pp} \), by direct integration

\[
I_\Box(x, \varepsilon) = \frac{2\kappa}{k_\vartheta} \left\{ \begin{pmatrix} v & 0 & 0 \\ 0 & d_2 & d_3 \\ 0 & d_3 & -d_2 \end{pmatrix} - \frac{r}{d_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-v \end{pmatrix} \right\} \left| \begin{array}{c} d_3 = b d_2 + k_\vartheta, \vartheta = b \\ d_2 = \sqrt{2} - x_2 \end{array} \right| + I_2 \frac{1}{\varepsilon} + o(1), \quad \varepsilon \to 0
\]
where
\[
\kappa = \frac{G}{8\pi (1-\nu)}, \quad I_2 = 4\kappa \sqrt{2} \begin{pmatrix}
2 & 0 & 0 \\
0 & 2-\nu & 0 \\
0 & 0 & 2-\nu
\end{pmatrix}
\] (63)

The finite part immediately follows from its definition. The same result follows from a limit process, by taking \(d_1 \to 0^+\) in Equations (60, 61). Considering only the term pertaining to a constant shape function, it holds in fact
\[
\lim_{d_1 \to 0^+} \mathbb{L}^{pp}_{\square} = \lim_{d_1 \to 0^+} \mathbb{R}^{pp}_{\square} = \lim_{d_1 \to 0^+} \mathbb{H}^{pp} = 0
\]
\[
\lim_{d_1 \to 0^+} \mathbb{S}^{pp}_{\square} = \frac{2\nu}{k\vartheta} \begin{pmatrix}
0 & 0 & 0 \\
0 & d_2 & d_3 \\
0 & d_3 & -d_2
\end{pmatrix}
\]

With the same path of reasoning, by direct integration it follows for the potential kernel \(G_{pp}^{\text{lapl}}\):
\[
I_{\square}^{\text{lapl}}(x, \vartheta) = \frac{\kappa}{k\vartheta} \frac{r}{d_2} \left| \begin{array}{c}
d_3 = bd_2 + k_b, \vartheta = b \\
d_3 = ad_2 + k_a, \vartheta = a \\
d_2 = -x_2
\end{array} \right| \left( d_2 = y_2 - x_2 \right) + \frac{\sqrt{2}}{\varepsilon} + o(1), \quad \varepsilon \to 0
\] (64)

and the finite part immediately follows from its definition. The same result follows from a limit process, by taking \(d_1 \to 0^+\) in Equations (60, 61). It holds in fact
\[
\lim_{d_1 \to 0^+} \mathbb{L}^{qq}_{\square} = \lim_{d_1 \to 0^+} \mathbb{R}^{qq}_{\square} = \lim_{d_1 \to 0^+} \mathbb{H}^{qq} = \lim_{d_1 \to 0^+} \mathbb{S}^{qq}_{\square} = \lim_{d_1 \to 0^+} \mathbb{H}^{qq} = 0
\]
\[
\lim_{d_1 \to 0^+} \mathbb{R}^{qq} = \frac{1}{d_2 k\vartheta}
\]

3.2.2. Cauchy’s principal value. Considerations about the nature of the singularity in the boundary limit apply to the CPV as well. Consider first the point \(x \in \Gamma\), so that kernel \(G_{pu}\) in the local coordinate system \(\mathcal{L}\) simplifies as
\[
G_{pu}(d; e_1) = \frac{1}{4\pi} \left( \frac{1-\nu}{r^3} \right) \text{SKW}(d \otimes e_1)
\]

By direct integration
\[
\int_{\mathcal{T}_j \setminus \mathcal{T}_f} G_{pu}(d; e_1) \, d\Gamma = 2\kappa \left[ \alpha \arctanh \frac{d_3}{r} - \frac{\beta}{\sqrt{1+\vartheta^2}} \log(\zeta_2 + r) \right]_{\begin{array}{c}
d_3 = bd_2 + k_b, \vartheta = b \\
d_3 = ad_2 + k_a, \vartheta = a \\
d_2 = -x_2
\end{array}} + o(1), \quad \varepsilon \to 0
\] (65)
where
\[ \kappa = \frac{1 - 2v}{8 \pi (1 - v)}, \quad \alpha = \text{SKW}(e_1 \otimes e_2), \quad \beta = \partial x - \text{SKW}(e_1 \otimes e_3) \]
which is the CPV by definition. Identity (65) follows even through a limit process, by taking \( d_1 \to 0^+ \) in (52). Considering only the terms pertaining to a constant shape function, it holds in fact
\[
\lim_{d_1 \to 0^+} \begin{pmatrix}
0 & -B & 1 \\
b & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} \frac{1 - 2v}{\sqrt{1 + b^2}}
\]
\[
\lim_{d_1 \to 0^+} \beta_{pu}^{pu} = 2(1 - 2v)\text{SKW}(e_1 \otimes e_2)
\]
\[
\lim_{d_1 \to 0^+} \beta_{pu}^{pu} = \lim_{d_1 \to 0^+} \beta_{pu}^{0u} = 0
\]
Strongly singular kernels \( G_{up} \) and \( G_{pu} \) generate free-terms [13] that holds for smooth boundaries in the limit process \( \Omega \ni x \to x \in \Gamma \). Such free-terms arise in the limit
\[
\lim_{d_1 \to 0^+} \left[ \beta_{pu}^{pu} (x, d_2, d_3) \right]_{d_3 = ad_2 + k_a} \Big|_{d_2 = \bar{y}_2 - x_2} {d_3 = bd_2 + k_b} \Big|_{d_2 = -x_2} = 1/2 \left[ \begin{array}{c}
1 \\
0
\end{array} \right] (66)
\]
In fact, taking into account the expansion
\[
\beta_{pu}^{pu} = 2d_1(1 - v) + o(d_1), \quad d_1 \to 0
\]
it can be easily shown that
\[
\kappa \lim_{d_1 \to 0^+} \left[ \beta_{pu}^{pu} (x, d_2, d_3) \right]_{d_3 = ad_2 + k_a} \Big|_{d_2 = \bar{y}_2 - x_2} {d_3 = bd_2 + k_b} \Big|_{d_2 = -x_2} = 1/2 \left[ \begin{array}{c}
1 \\
0
\end{array} \right] (67)
\]
By inserting outcome (67) into Equation (22), it follows that
\[
\frac{1}{2} \left[ \begin{array}{c}
1 \\
0
\end{array} \right] \left[ \begin{array}{c}
\frac{1 - x_2}{\bar{y}_2} \\
\frac{1}{\bar{y}_2}
\end{array} \right] = \frac{1}{2} \theta_j^p(x)
\]
which is the discrete counterpart of the free-term for smooth boundaries.

4. DISCUSSION

4.1. Literature review

Analytical integrations have been basically performed toward three schemes. In the first scheme (see e.g. [14, 31]), the source point is fixed, whereas the boundary around the source point is temporarily deformed to allow an analytical evaluation of contributions from singular kernels; then
4.2. On function $I_{\Delta}^{r^{-3}}(\mathbf{x})$ and its implementation

A preliminary work [26] concerned the analytical integration of function $r^{-3}$ over triangle $T_j$, as the sum of two factors $I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2)$:

$$I_{\Delta}^{r^{-3}}(\mathbf{x}) = \int_{-x_2}^{\bar{y}_2-x_2} \int_{ad_2+k_b}^{bd_2+k_b} \frac{1}{r^3} \, dd_2 \, dd_3 = I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2) \bigg|_{-x_2}^{\bar{y}_2-x_2}$$  \hspace{1cm} (68)

where $I_{\Delta}^{r^{-3}}(\ldots) : [\mathbb{R}^3 \setminus T_j \times [-x_2, \bar{y}_2 - x_2]] \rightarrow \mathbb{R}$ was defined by

$$I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2) = \int \, dd_2 \, \int_{ad_2+k_b}^{bd_2+k_b} \frac{1}{r^3} \, dd_3 = I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2, d_3) \bigg|_{d_3=ad_2+k_b}^{d_3=bd_2+k_b}$$  \hspace{1cm} (69)

Focusing on the upper extremum $d_3 = bd_2+k_b$, and defining with

$$\Upsilon(\mathbf{x}, d_2, bd_2+k_b) = bd_1 + \frac{b(d_1^2+d_2^2)d_1}{2d_2^2} + \frac{(d_1^2+d_2^2)((d_1^2+d_2^2)b^2+d_2^2)d_1}{2d_2^2(d_2k_b-bd_1^2)} + \frac{(d_1^2-d_2^2)k_b}{2d_2d_1}$$

two candidate functions for $I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2, bd_2+k_b)$ arise, namely

$$f_1^b(\mathbf{x}, d_2, bd_2+k_b) = \frac{1}{2d_1} \arctan \frac{r}{\Upsilon(\mathbf{x}, d_2, bd_2+k_b)}$$

$$f_2^b(\mathbf{x}, d_2, bd_2+k_b) = -\frac{1}{2d_1} \arctan \frac{\Upsilon(\mathbf{x}, d_2, bd_2+k_b)}{r}$$

which are linked by

$$f_1^b - f_2^b = \frac{\pi}{4d_1} \text{sgn} \frac{\Upsilon(\mathbf{x}, d_2, bd_2+k_b)}{r}$$  \hspace{1cm} (70)

The (unique) function $I_{\Delta}^{r^{-3}}(\mathbf{x}, d_2, bd_2+k_b)$ can be caught in studying the domain in which $f_1^b$ and $f_2^b$ are defined. Within a domain where both $f_1^b$ and $f_2^b$ are defined, they have the same derivative, but they differ by a constant. Within a domain in which only $f_1^b$ (or $f_2^b$) is everywhere defined, $f_1^b$ (or $f_2^b$) is the unique primitive. The analysis is quite involved: a flow chart summarizes it in Figure 4.
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4.3. Implementation details

The singularity analysis performed in Section 3.2 does not apply when \( x \in \partial T_j \); as it is well known, the limit to the boundary process looses any meaning, because a uniquely defined normal to the boundary cannot—in general—be identified on \( \partial T_j \).

4.3.1. Evaluation of matrices \( R^{pp}, S^{pp}, H^{pp} \). The evaluation of matrices \( R^{pp}, S^{pp}, H^{pp} \) requires particular care when point \( x \) belongs to the lines that define the sides of \( T_j \), even for \( x \not\in \partial T_j \).

In view of definitions (A1) and (A7), in the analysis of matrices \( R^{pp}, S^{pp} \) for potential problems three issues shall be investigated, namely: \( \partial^2 d_1^2 + k_2^2 = 0 \), \((1 + \partial^2) d_1^2 + k_2^2 = 0 \), and \( d_1^2 + d_2^2 = 0 \). Their meaning is summarized in Figure 5.

Figure 4. A flow chart of \( f^b(x, d_2) \).

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Figure 5. The three special cases $\vartheta^2d_1^2 + k_2^2 = 0$, $(1 + \vartheta^2)d_1^2 + k_2^2 = 0$ and $d_1^2 + d_2^2 = 0$ and their geometrical meaning.

is well defined however when $\mathbf{x} \notin \partial T_j$. Noting that $-d_2n_3/d_1$ is the constant term of the expansion of $\mathbb{S}^{pp}\cdot\mathbf{e}_1$ about $\vartheta = 0$, algorithm (61) can be kept as it stands by assuming

$$\mathbb{S}^{pp}\cdot\mathbf{e}_1 = \begin{cases} \lambda(\vartheta)\Lambda(\vartheta)d_1\mathbf{n} \cdot \mathbf{a}, & \vartheta \neq 0 \text{ or } k_{\vartheta} \neq 0 \\ \frac{d_2n_3}{d_1}, & x_1 \neq 0 \\ 0, & x_1 = 0, x_2 < 0 \text{ or } x_2 > \bar{y}_2 \end{cases}$$

$$\mathbb{R}^{pp}\cdot\mathbf{e}_1 = \begin{cases} \frac{\lambda(\vartheta)}{(d_1^2 + d_2^2)}\mathbf{n} \cdot \mathbf{c}, & \vartheta \neq 0 \text{ or } k_{\vartheta} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Similar remarks apply to the linear term in algorithm (61) that can be kept as it stands by assuming

$$\mathbb{S}^{pp}\cdot\mathbf{e}_2 = \begin{cases} \lambda(\vartheta)\Lambda(\vartheta)d_1\mathbf{n} \cdot \mathbf{a}, & \vartheta \neq 0 \text{ or } k_{\vartheta} \neq 0 \\ -d_1n_3, & x_1 \neq 0 \\ 0, & x_1 = 0, x_2 < 0 \text{ or } x_2 > \bar{y}_2 \end{cases}$$

$$\mathbb{R}^{pp}\cdot\mathbf{e}_2 = \begin{cases} \frac{\lambda(\vartheta)}{(d_1^2 + d_2^2)}\mathbf{n} \cdot \mathbf{c}, & \vartheta \neq 0 \text{ or } k_{\vartheta} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Vectors $\mathbf{a}$ and $\gamma$ have been defined in Appendix C.

The two equalities $d_1 = k_{\vartheta} = 0$ identify two sides of $T_j$. Neither $\mathbb{R}^{pp}$ nor $\mathbb{S}^{pp}$ are defined, but the sum

$$\mathbb{R}^{pp}r + \mathbb{S}^{pp}\frac{\mathbf{e}_1}{r}d_2 = \vartheta_n \frac{1}{r}d_2 = \vartheta_n \frac{1}{r}d_2 = \bar{y}_2 - x_2, \quad x_2 < 0 \text{ or } x_2 > \bar{y}_2$$

(72)

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is well defined, however when \( \mathbf{x} \notin \partial T_j \). Algorithm (61) can be kept as it stands by assuming

\[
\mathbb{S}^{pp} \cdot \mathbf{e}_1 = \begin{cases} 
\lambda(\partial) \Lambda(\partial) d_1 \mathbf{n} \cdot \mathbf{z}, & d_1 \neq 0 \quad \text{or} \quad k_\partial \neq 0 \\
-\vartheta n_1, & x_2 < 0 \quad \text{or} \quad x_2 > y_2 
\end{cases}
\]

\[
\mathbb{R}^{pp} \cdot \mathbf{e}_1 = \begin{cases} 
\frac{\lambda(\partial)}{(d_1^2 + d_2^2)} \mathbf{n} \cdot \mathbf{\gamma}, & \vartheta \neq 0 \quad \text{or} \quad k_\partial \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]

The linear term does not require particular care.

The last case, \( d_1 = d_2 = 0 \), concerns the side of \( T_j \) orthogonal to plane \( y_3 = 0 \). Vector \( \mathbb{S}^{pp} \) is well defined (it is vanishing indeed), whereas \( \mathbb{R}^{pp} \) is not. The quantity

\[
\mathbb{R}^{pp} \bigg|_{\vartheta = b; \vartheta = a} \cdot \mathbf{e}_1 = \frac{\vartheta n_1}{k_\partial} \bigg|_{\vartheta = a} \frac{x_3}{y_2} < a \quad \text{or} \quad \frac{x_3}{y_2} > b
\]

is well defined, however when \( \mathbf{x} \notin \partial T_j \). Algorithm (61) can be kept as it stands by assuming

\[
\mathbb{R}^{pp} \cdot \mathbf{e}_1 = \begin{cases} 
\frac{\lambda(\partial)}{(d_1^2 + d_2^2)} \mathbf{n} \cdot \mathbf{\gamma}, & d_1 \neq 0 \quad \text{or} \quad d_2 \neq 0 \\
\frac{\vartheta n_1}{k_\partial}, & x_3 < a \quad \text{or} \quad \frac{x_3}{y_2} > b
\end{cases}
\]

The linear term does not require particular care.

4.3.2. Evaluation of \( \text{arctanh}(d_3/r) \). Function \( \text{arctanh}(d_3/r) \) is not defined when \( |d_3/r| = 1 \), that is when \( d_1 = d_2 = 0 \). This last condition defines the line of the side of \( T_j \) parallel to axis \( y_3 \) and its parallel through the center of the coordinate system (see Figure 5). It can be seen from Figure 6 that \( \mathbf{x} \notin \partial T_j \) implies \( \text{sgn}(k_a) = \text{sgn}(k_b) \). By exploiting this property, the asymptotical analysis about \( d_1 = d_2 = 0 \) shows that in such a case \( \text{arctanh}(d_3/r) \) must be replaced by \( \text{sgn}(k_3) \log |k_3| \) in outcomes (44), (51), (52) and (61).

4.3.3. Evaluation of \( \log(\zeta_2 + r) \). In view of definition (39), function \( \log(\zeta_2 + r) \) is not appropriately defined for \( d_1 = k_\partial = 0 \). Such a condition defines the lines of the two sides of \( T_j \) emanating from the center of the coordinate system (see Figure 5). It can be seen from Figure 6 that \( \mathbf{x} \notin \partial T_j \) implies \( \text{sgn}(d_2) \) is constant for any \( d_3 \). By exploiting this property, the asymptotical analysis about \( d_1 = k_\partial = 0 \) shows that in such a case \( \log(\zeta_2 + r) \) must be replaced by \( \text{sgn}(d_2) \log |d_2| \) in outcomes (44), (51), (52), and (61).

4.4. On the computational cost of analytical integrations

The interest in analytical integrations is not limited to computational efficiency; nevertheless, as they in principle provide machine-precision accuracy, ‘measuring’ their computational cost is a primary concern. A key point to this aim is envisaging a unit of measurement independent of the computing machine, on the implemented code, on the programming language and also on the
A. SALVADORI

Figure 6. Different values of $d_2$ and $k_\phi$ depending on the position of point $x$.

Table II. A measure of the computational cost of the proposed analytical integrations for the Laplace equation. Indeed results for Lamé operator looks a bit better, because computational effort in (24) seems to be mainly due to scalar functions (log, arctan).

<table>
<thead>
<tr>
<th></th>
<th>$\mu(G_{uu})$</th>
<th>$\mu(G_{up})$</th>
<th>$\mu(G_{pu})$</th>
<th>$\mu(G_{pp})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ co-planar with $T_j$</td>
<td>6</td>
<td>—</td>
<td>—</td>
<td>6.16</td>
</tr>
<tr>
<td>$x$ out of plane</td>
<td>7.06</td>
<td>6.22</td>
<td>6.53</td>
<td>7.13</td>
</tr>
</tbody>
</table>

Such a measure does not depend on the hardware, but it does on the implementation. The results relevant to the Laplace equation are collected in Table II.

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Figure 7. Set of field points $\mathbf{x}$ for the iso-Gauss points maps of Figures 8 and 9. The set for the far field is made of points that belong to semi-circumferences of radius $r = 2^k r_0$, with $r_0$ the distance between the lower vertex and the incenter of the triangle; $k = 1, 2, 3, 4$. The near field point set is fully depicted.

Measure (74) can be significantly related to the number of nodes $n_{Q}(\mathbf{x})$ of a quadrature rule $Q_{rs}(\mathbf{x})$ eventually used to approximate integral (22). Comparisons must be made with respect to a target accuracy

$$\left\| \frac{Q_{rs}(\mathbf{x}) - K_{rs}(\mathbf{x})}{K_{rs}(\mathbf{x})} \right\| < \varepsilon$$

The higher the accuracy, the higher the $n_{Q}(\mathbf{x})$, obviously. The number of nodes $n_{Gauss}(\mathbf{x})$ of a Gaussian standard quadrature rule

$$\int_{a}^{b} \int_{a}^{b} g(y_2, y_3) \, dy_3 \, dy_2 \approx \frac{2^2}{8} (b-a) \sum_{k=1}^{n_{Gauss}} w_k (1 + x_k) \sum_{i=1}^{n_{Gauss}} w_i g(y_2(x_k), y_3(x_i, x_k))$$

is depicted in Figures 8 and 9. They map $n_{Gauss}(\mathbf{x})$ required to reach accuracy $\varepsilon = 10^{-6}$ at field point $\mathbf{x}$. The set of nodes where kernels $G_{uu}$ and $G_{pp}$ have been evaluated is plotted in Figure 7. The panel $T_j$ of integral (22) is depicted as well.

In terms of efficiency, analytical integrations are comparable to a Gauss quadrature in the far field, whereas the usage of analytical formulae seems to be useful in the near-field. Additional work will take into account the several quadrature rules—some of which are extremely recent [39]—available in the literature.

4.5. Benchmarks

The following benchmarks aim at showing the capability of the analytical formulation proposed. They pertain to the ‘hypersingular collocation’ scheme, whose limitation and drawbacks are well known; the goal is merely to show the effectiveness of the integration strategy, not of the numerical scheme: the analysis and the implementation of the Galerkin approach is in progress and will be considered in a forthcoming publication.
Figure 8. Number of nodes \( n_{\text{Gauss}}(x) \) required to reach accuracy \( \zeta = 10^{-6} \) for a Gauss rule for the quadrature of kernel \( G_{uu} \) for the Laplace operator. Pictures (a) and (b) refer to the plane \( y_1 = 0 \) and might be compared with the first row of Table II. The picture is plotted exploiting symmetry of the involved fields about axes \( y_3 = 0 \). Pictures (c) and (d) refer to the plane \( y_3 = 0 \) and might be compared with the second row of Table II. Contour line crosses the triangle in Figure (b) because of the interpolation algorithm: indeed \( n_{\text{Gauss}}(x) \) has been evaluated onto the set of points of Figure 7.

**Laplace operator on a cube:** Consider the harmonic function \( f(x) = x_3(x_1^2 - x_2^2) \). An approximation of \( f(x) \) is sought for on a cube of side \( L = 20 \) centered at the origin of the coordinate system, assuming linear shape functions for potential and fluxes. A mixed boundary value problem is set up, by constraining one quarter of a cube side. Four different meshes, recursively obtained by the Sierpinsky map on the cube sides, are described in Figure 10; as a property of the Sierpinsky map, the radius of the circumcircle, assumed as triangulation parameter, geometrically decreases with the recursion index \( h = L / 2^h \) and the number of elements is related to \( h \) as \( E_1 = (24/L^2)h^2 \).

Accuracy of the scheme is measured in several norms. Displacement and gradients have been evaluated at a set of points \( P_j = \{j, 0, j\} \), \( j = 1, 2, \ldots, 9 \) and errors \( \varepsilon_j^u = ||u_h(P_j) - f(P_j)||/||f(P_j)|| \)
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Figure 9. Number of nodes $n_{\text{Gauss}}(\mathbf{x})$ required to reach accuracy $\zeta = 10^{-6}$ for a Gauss rule for the quadrature of kernel $G_{pp}$ for the Laplace operator. Pictures (a) and (b) refer to the plane $y_1 = 0$ and might be compared with the first row of Table II. The picture is plotted exploiting symmetry of the involved fields about axes $y_3 = 0$. Pictures (c) and (d) refer to the plane $y_3 = 0$ and might be compared with the second row of Table II.

for potential and $\varepsilon^p_j = \| p_h(P_j) - z \nabla f(P_j) \| / \| z \nabla f(P_j) \|$ for gradient** are plotted in Figure 11; values for $j = 1, 5, 9$ are tabulated in Tables III and IV. Convergence is envisaged, but its rate appears to be hardly measurable from $\varepsilon^u_j$ and $\varepsilon^p_j$.

The relative error in $L_2$ norm for the Dirichlet and the Neumann data

$$
\varepsilon^u_{L_2} = \frac{\| u_h(\mathbf{x}) - f(\mathbf{x}) \|_{L_2(\Gamma_p)}}{\| f(\mathbf{x}) \|_{L_2(\Gamma_p)}}, \quad \varepsilon^p_{L_2} = \frac{\| p_h(\mathbf{x}) - z \nabla f(\mathbf{x}) \cdot \mathbf{n} \|_{L_2(\Gamma_u)}}{\| z \nabla f(\mathbf{x}) \cdot \mathbf{n} \|_{L_2(\Gamma_u)}}
$$

(75)

is reported in Table V together with error $\varepsilon_E$ in bilinear form (8). Denoting with $y^T = [ f(\mathbf{x}), z \nabla f(\mathbf{x}) \cdot \mathbf{n} ]$ the exact solution and with $y_h$ the hypersingular collocation approximation,

**Here $\| g \| = \sqrt{g \cdot g}$ and $z$ is the material constitutive parameter that relates fluxes to potential (e.g. conductivity for heat transfer).
Figure 10. Four different meshes, recursively obtained with the Sierpinsky application on the cube sides. Further data are printed in Table III.

Figure 11. A wider spectra for the gradient error $\varepsilon_j^p = | p_h(P_j) - \alpha \nabla f(P_j) | | \alpha \nabla f(P_j) |$ with respect to Table IV.

Table III. Comparison of different meshes. rec stands for the recursion index, $N$ for the number of nodes, El for the number of elements, $h$ for the radius of elements circumcircle, $L$ for cube side (in the analysis taken as $L=20$), $\varepsilon_j$ for the relative error at point $P_j$ and $CR_j$ is the convergence rate at point $P_j$.

<table>
<thead>
<tr>
<th>rec</th>
<th>N</th>
<th>El</th>
<th>$h$</th>
<th>$\varepsilon_{\text{u}}^1$</th>
<th>CR$_1$</th>
<th>$\varepsilon_{\text{u}}^5$</th>
<th>CR$_5$</th>
<th>$\varepsilon_{\text{u}}^9$</th>
<th>CR$_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14</td>
<td>24</td>
<td>$L$</td>
<td>5.43E$-1$</td>
<td></td>
<td>6.34E$-1$</td>
<td></td>
<td>7.24E$-1$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>96</td>
<td>$L/2$</td>
<td>4.69E$+0$</td>
<td>0.12</td>
<td>6.21E$-2$</td>
<td>10.21</td>
<td>2.28E$-3$</td>
<td>317.13</td>
</tr>
<tr>
<td>3</td>
<td>194</td>
<td>384</td>
<td>$L/4$</td>
<td>8.05E$-1$</td>
<td>5.82</td>
<td>1.59E$-2$</td>
<td>3.92</td>
<td>1.19E$-2$</td>
<td>0.19</td>
</tr>
<tr>
<td>4</td>
<td>770</td>
<td>1536</td>
<td>$L/8$</td>
<td>1.42E$-1$</td>
<td>5.68</td>
<td>9.27E$-3$</td>
<td>1.71</td>
<td>7.28E$-3$</td>
<td>1.63</td>
</tr>
</tbody>
</table>

Note that $O(\text{El}) = O(h^2)$. The displacement error measured by $\varepsilon_j^u$ is also reported.
which is a solution of (77). The stress tensor relevant to (78) reads as

\[ \sigma = \frac{E}{1-2v} \mathbb{I} \]

where \( \mathbb{I} \) denotes the identity matrix, \( E \) is Young’s modulus and \( v \) is the Poisson coefficient. Traction vector at point \( x \) acting on a surface of normal \( n(x) \) is given by

\[ p(x, n(x)) = \sigma(x) n(x) = \frac{E}{1-2v} n(x) \]

A mixed boundary value problem has been set imposing (78) as a Dirichlet boundary condition on boundary \( \Gamma_u \subseteq \Gamma \) and (80) as a Neumann boundary condition on boundary \( \Gamma_p \subseteq \Gamma \); boundary \( \Gamma \) is taken such that \( \Gamma_u \cap \Gamma_p = \Gamma \) and \( \Gamma_u \cap \Gamma_p = \emptyset \). Several combinations of \( \Gamma_u \) and \( \Gamma_p \) have been tested.

\( \varepsilon_E \) is defined as

\[ \varepsilon_E = \mathcal{A}_{\mathcal{G}}((y-y_h), y) = \frac{\int_{\Gamma_u} (p_h(x) - \alpha \nabla f(x) \cdot n) f(x) \, d\Gamma + \int_{\Gamma_p} (u_h(x) - f(x)) \alpha \nabla f(x) \cdot n \, d\Gamma}{\int_{\Omega} f(x) \alpha \nabla f(x) \cdot n \, d\Gamma} \]  

It is worth noting that bilinear form (8) is not positive definite, but that \( \mathcal{A}_{\mathcal{G}}(y, y) \geq 0 \), what comes out also from the internal energy meaning of \( \mathcal{A}_{\mathcal{G}} \) at problem solution \( y \).

**Lamé operator on an icosahedron**: This benchmark is devoted to Navier’s equation:

\[ G \Delta u(x) + (\lambda + G) \nabla \text{div} u(x) = 0, \quad x \in \Omega \]  

with \( G \) and \( \lambda \) denoting the Lamé constants. They are taken as independent of \( x \), namely material is taken as homogeneous in domain \( \Omega \). The latter is assumed to be an icosahedron.

Consider displacement field

\[ u(x) = x \]

which is a solution of (77). The stress tensor relevant to (78) reads as

\[ \sigma(x) = \frac{E}{1-2v} \mathbb{I} \]  

\begin{table}[ht]
\centering
\caption{The gradient error measured by \( \varepsilon_{L^2}^u \), \( \varepsilon_{L^2}^p \) and \( \varepsilon_E \).}
\begin{tabular}{cccccccc}
\hline
\text{rec} & \text{N} & \text{El} & \text{h} & \text{\( \varepsilon_{L^2}^u \)} & \text{CR}_{\text{u,L}^2} & \text{\( \varepsilon_{L^2}^p \)} & \text{CR}_{\text{p,L}^2} & \text{\( \varepsilon_E \)} & \text{CR}_{\text{E}} \\
\hline
1 & 14 & 24 & \text{L} & 1.00E+00 & & 2.60E−01 & & 1.00E+00 & \\
2 & 50 & 96 & \text{L/2} & 1.96E−01 & 5.10 & 5.40E−02 & 4.81 & 1.41E−01 & 7.11 \\
3 & 194 & 384 & \text{L/4} & 4.60E−02 & 4.26 & 1.90E−02 & 2.84 & 3.11E−02 & 4.52 \\
4 & 770 & 1536 & \text{L/8} & 1.40E−02 & 3.29 & 9.20E−03 & 2.07 & 1.16E−02 & 2.68 \\
\hline
\end{tabular}
\end{table}

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It is expected that the BEM approximation by means of linear shape functions recovers the exact solution, because integrations have been analytically solved. In all simulations that have been performed, the numerical approximation showed an accuracy of at least 13 digits.

The Lamé benchmark has been made less trivial by considering different materials (see Figure 12(a)). Each material has been taken as homogeneous, with constants:

\[ v_n = 0.1n, \quad E_n = 1 - 2v_n, \quad n = 1, 2, 3 \]  

(81)

The stress tensor relevant to displacement field

\[ \mathbf{u}_n(x) = x \]  

(82)

amounts to the identity matrix for all \( n = 1, 2, 3 \). Therefore, traction vector relevant to (82) at point \( x \) acting on a surface of normal \( \mathbf{n}(x) \) is given by \( \mathbf{p}_n(x, \mathbf{n}(x)) = \mathbf{n}(x) \) for all \( n = 1, 2, 3 \).

A mixed boundary value problem has been set up imposing (82) as a Dirichlet boundary condition together with \( \mathbf{p}_n(x, \mathbf{n}(x)) = \mathbf{n}(x) \) as a Neumann boundary condition. The multizone boundary integral problem has been formulated according to \[40\] and solved. Three isotropic bodies \( \Omega_n \) have been rigidly connected, i.e. neither opening nor sliding has been allowed at the interfaces. Two of them are depicted in Figure 12(b), whereas the third covers the excluded volume. Several combinations of \( \Gamma_u \) and \( \Gamma_p \) have been tested.

In all simulations that have been performed, the numerical approximation of \( \mathbf{u}_n(x) = x \) at several field points \( x \in \Omega_n \) showed an accuracy higher than 13 digits.

4.6. Final remarks

In the present note, analytical integrations have been performed for BIEs (4)–(5) in their discrete form; both the singular and the regular part have been considered, so that the closed form is obtained as a function of the collocation point. The proposed outcomes are exhaustive for the collocation approach as well as for the post-process reconstruction of primal and dual fields (temperature and flux, displacement and stress). It seems to be of interest for the Galerkin technique as well, because it firmly distinguishes all the terms (emphasis must be given on the weakly singular ones) relevant to the outer integral; the approach pursued in [36] maps the road to show the mutual cancelation of the singular terms in the outer integration process. Analytical integration for static (steady-state) problems is also the main ingredient for the evaluation of closed forms of integrals.
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pertaining to time-dependent problems, such as elastodynamics and acoustics, which have been recently considered [41].

The availability of the closed form for the approximated primal and dual fields entails the possibility of analytical manipulations—see e.g. [42]. Indeed, closed forms (44), (51), (52) and (61) allow the extension to 3D fracture mechanics of the important result of Gray and Paulino [43]: in a nutshell, it has been shown that—as it happens in two dimensions—the linear term of the expansion of crack opening and sliding about the crack tip vanishes. The argument of the proof is that a linear term in such an expansion induces a logarithmic singularity for stresses at the crack front that is not compatible with the asymptotical behavior of the stress field. Details will be published in a further publication.

5. CONCLUSIONS

Analytical evaluation of ‘integrals’ pertaining to the BEM has been provided. It is relevant to the collocation BEM technique (11) as well as to the inner integral of symmetric Galerkin BEM (12), after performing a tessellation of the boundary by means of flat triangles. Shape functions $\phi_n(y)$ have been taken as polynomials of arbitrary degree: this allows—in the non-unusual case of flat surfaces (see e.g. [44])—a $p$-refinement technique.

Integrations have been performed in a local coordinate system, which was detailed in Section 2.1. All significant instances of the position of the source point have been analyzed. In particular, when point $x$ belongs to triangle $T_j$, the integral does not exist in a classical sense. In such cases, the continuity (with respect to the source point) between the HFP and the Lebesgue integral has been shown. To this aim, the HFP has been directly evaluated as first; further, the limit process to the boundary $\Omega \ni x \to x \in \Gamma$ has been performed.

The implementation requires particular care in some pathological configurations. Section 4.3 has been devoted at describing code development details. Some benchmarks have been proposed in Section 4.5, with the mere aim of verifying the capabilities of the proposed analytical integrations and their efficiency.

Broader impacts may result from the present note: on the educational side, introductory courses in the BIEs may benefit from the closed form for Equations (4)–(5), which may lighten the burden of numerical approximation of elliptic problems via BEM to inexpert auditors.

APPENDIX A: GREEN’S FUNCTIONS

The expressions of Green’s functions for 3D Laplace and linear elasticity follow. Here $n(x)$ and $l(y)$ are the normals at the boundary at $x$ and $y$, respectively. Vectors $d$ and $r$ are defined as $d = -r = (y - x)$.

A.1. Laplace equation

$$G_{uu}(r) = \frac{1}{4\pi r} \frac{1}{r}$$

$$G_{pu}(r; n(x)) = -\frac{1}{4\pi r} \frac{r \cdot n}{r^3}$$

A.2. Linear elasticity

\[ G_{up}(d; l(y)) = \frac{1}{16\pi} \frac{1}{G(1-v)} \frac{1}{r} \left( \frac{d \otimes d}{r^2} + (3-4v)I \right) \]

\[ G_{pp}(r; n(x); l(y)) = -\frac{1}{4\pi r^3} \left( \frac{(r \cdot l)(r \cdot n)}{r^2} - n \cdot l \right) \]

APPENDIX B: FUNDAMENTAL LEBESGUE INTEGRALS

The following identities, which can be proved by induction when \( d_1 \neq 0 \), are the keynote of the inner integration. Here, the following notation will be considered:

*\( \hat{k} = k \div 2 \) integer division \( k \div 2 \).

*\( k_{[2]} = k - 2\hat{k} \) remainder of the (integer) division \( k \div 2 \).

*\( I_{\Delta}^r(x, d_2, d_3) \) has been defined in Section 4.2.
ANALYTICAL INTEGRATIONS IN 3D BEM

Proposition
By defining with

\[ \lambda(\vartheta) = \frac{1}{\vartheta^2 d_1^2 + k_\vartheta r^2}, \quad d_3 = \vartheta d_2 + k_\vartheta \]  \hspace{1cm} (A1)

it holds

\[ \int_{-x_2}^{y_2-x_2} \frac{d_2^{[2]}(\vartheta)}{d_1^2 + d_2^2} \frac{1}{r} d\vartheta = \lambda(\vartheta) \left[ \vartheta \arctanh \frac{d_3}{r} + \beta I_{\Delta}^{-3} (x, d_2, d_3) \right]_{\vartheta=a}^{\vartheta=b} d_2 = y_2 - x_2 \]  \hspace{1cm} (A2)

\[ \int_{-x_2}^{y_2-x_2} \frac{d_2^{[2]}(\vartheta)}{(d_1^2 + d_2^2 r^2)^2} \frac{1}{r} d\vartheta = \lambda^3(\vartheta) \left[ \vartheta \arctanh \frac{d_3}{r} + \delta I_{\Delta}^{-3} (x, d_2, d_3) + \zeta r \right]_{\vartheta=a}^{\vartheta=b} d_2 = y_2 - x_2 \]  \hspace{1cm} (A3)

\[ \int_{-x_2}^{y_2-x_2} \frac{d_2^{[2]}(\vartheta)}{(d_1^2 + d_2^2 r^2)^3} \frac{1}{r} d\vartheta = \lambda^5(\vartheta) \left[ \vartheta \arctanh \frac{d_3}{r} + \theta I_{\Delta}^{-3} (x, d_2, d_3) + \eta r \right]_{\vartheta=a}^{\vartheta=b} d_2 = y_2 - x_2 \]  \hspace{1cm} (A4)

where

\[ \vartheta = -k_\vartheta n_{[2]} + \vartheta(1 - n_{[2]}) \]

\[ \beta = \vartheta d_2^2 n_{[2]} + k_\vartheta (1 - n_{[2]}) \]

\[ \gamma = \frac{k_\vartheta}{2} (4k_\vartheta^2 - 3 + 2 \vartheta^2) n_{[2]} + \frac{\vartheta}{2} \left( \frac{1 + 2 \vartheta^2}{\lambda(\vartheta)} - 4k_\vartheta^2 \right) (1 - n_{[2]}) \]

\[ \delta = \frac{d_2^2 \big[ d_1^2 \vartheta^2 (1 + \vartheta^2) - 3k_\vartheta^2 \big] - k_\vartheta^4}{2n_{[2]} + k_\vartheta d_1^2 (3 \vartheta^2 d_1^2 (1 + \vartheta^2) - k_\vartheta^2 (1 - 4 \vartheta^2)) + k_\vartheta^4 (1 - n_{[2]})} \]

\[ \zeta = \frac{\vartheta^2 d_2^2 + 2 \vartheta d_2 k_\vartheta - k_\vartheta^2}{2(d_1^2 + d_2^2) \lambda(\vartheta)} n_{[2]} - \frac{\vartheta d_2^2 (\vartheta d_2 - 2 k_\vartheta)}{2d_1^2 (d_1^2 + d_2^2) \lambda(\vartheta)} (1 - n_{[2]}) \]

\[ \eta = \frac{k_\vartheta}{8} \left( -8d_2^2 \vartheta^6 - 8(3d_1^2 + k_\vartheta^2) \vartheta^4 - 3(5d_1^2 - 8k_\vartheta^2) \vartheta^2 + 45k_\vartheta^2 \right) n_{[2]} \]

\[ + \frac{\vartheta}{8} \left( 8d_2^2 \vartheta^6 + 8(d_1^2 + k_\vartheta^2) \vartheta^4 + (3d_1^2 - 40k_\vartheta^2) \vartheta^2 - 33k_\vartheta^2 \right) \lambda(\vartheta) + 48k_\vartheta^4 \]  \hspace{1cm} (1 - n_{[2]})

\[ \theta = \frac{\vartheta d_1}{8} (3 \vartheta^8 d_1^5 + 6 \vartheta^6 d_1^5 + \vartheta^4 d_1^3(3d_1^4 - 30d_1^2 k_\vartheta^2 - 10k_\vartheta^4)) \]

\[ + \frac{2 \vartheta^2 d_1^5 (15d_1^4 k_\vartheta^2 + 15d_1^2 k_\vartheta^4 + 4k_\vartheta^6) + k_\vartheta^4 (15d_1^4 + 6d_1^2 k_\vartheta^2 - k_\vartheta^4)}{d_1^3} \]  \hspace{1cm} n_{[2]}
\[ + \frac{k \delta d_1}{8} (15 \theta^6 d_1^5 + 10 \theta^6 d_1^3 (3d_1^2 + 4k_\theta^2) + \theta^4 d_1 (15d_1^4 + 10d_1^2 k_\theta^2 + 38k_\theta^4) \\
+ \frac{\theta^2 d_1^5 (-30d_1^4 k_\theta^2 - 22d_1^2 k_\theta^4 + 16k_\theta^6) + k_\theta^2 (3d_1^4 - 2d_1^2 k_\theta^2 + 3k_\theta^4)}{d_1^5} (1 - n_{[2]})) \\
+ \frac{\theta^2 d_1^2 + 2 \theta d_2 k_\theta - k_\theta^2}{4(d_1^2 + d_2^2)^2 \lambda^3 (\theta) n_{[2]} + \frac{-\theta^2 d_1^2 d_2 + 2 \theta d_2^2 k_\theta + d_2 k_\theta^2}{4d_1^2 (d_1^2 + d_2^2)^2 \lambda^3 (\theta) (1 - n_{[2]})} \\
+ \{3 \theta^6 d_1^6 + 11 \theta^5 d_1^4 d_2 k_\theta + \theta^4 (3d_1^6 - 6d_1^4 k_\theta^2) + \theta^3 (12d_1^4 d_2 k_\theta + 10d_1^2 d_2 k_\theta^3) \\
+ \theta^2 (18d_1^4 d_2 k_\theta^2 - 9d_1^2 k_\theta^4) + \theta (12d_1^2 d_2 k_\theta^3 - d_2 k_\theta^5) + 3d_1^4 k_\theta^4 \} \frac{1}{8d_1^2 (d_1^2 + d_2^2) \lambda (\theta) n_{[2]} \\
+ \{ -6 \theta^6 d_1^6 d_2 + 13 \theta^5 d_1^4 d_2 k_\theta + \theta^4 (-3d_1^6 d_2 + 3d_1^4 d_2 k_\theta^2) + \theta^3 (12d_1^4 d_2 k_\theta + 14d_1^2 d_2 k_\theta^3) \\
+ \theta^2 (18d_1^4 d_2 k_\theta^2 + 12d_1^2 d_2 k_\theta^4) + \theta (-12d_1^4 k_\theta^3 + d_2 k_\theta^5 - 3d_1^4 d_2 k_\theta^3 + 3d_2 k_\theta^6) \} \frac{1 - n_{[2]}}{8d_1^4 (d_1^2 + d_2^2) \lambda (\theta) n_{[2]}} \] (A5)

**Proposition**

By defining with \( d_3 = \theta d_2 + k_\theta \) it holds

\[
\int_{-x_2}^y \frac{d_2^k \log (d_3 + r)}{r} \bigg|_{\theta=b} d_2 - \sum_{n=0}^k \sigma(n) \int_{-x_2}^y \frac{d_2^{k-n}}{r} d_2 = \left[ \mu \arctan \frac{d_5}{r} + v I_{\Delta}^{-3} (x, d_2, d_3) + \xi \log (d_3 + r) \right]_{\theta=a} \bigg|_{d_2=-x_2}^{d_2=y_2-x_2} \] (A6)

where

\[
\sigma(n) = (-1)^n \frac{d_1^n}{k+1} (k_b - (k_b + b d_1) n_{[2]}) \\
\xi = \frac{d_2^{k+1}}{k+1} \\
v = (-1)^k \frac{d_1^{k+2}}{k+1} (k_{[2]} - 1) \\
\mu = (-1)^k \frac{d_1^{k+1}}{k+1} k_{[2]} \\
\]

**Proposition**

With assumptions (A1) and

\[
\Lambda(\vartheta) = \frac{1}{d_1^2 (1 + \vartheta^2) + k_\vartheta^2} \] (A7)

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where

\[
\begin{align*}
\rho &= \lambda^2(\vartheta) k_\vartheta (3 \vartheta^2 d_1^2 - k_\vartheta^2) n_{[2]} + \lambda^2(\vartheta) \vartheta (-\vartheta^2 d_1^2 + 3 k_\vartheta^2)(1 - n_{[2]}) \\
q &= -\lambda^2(\vartheta) \vartheta d_1^2 (\vartheta^2 d_1^2 - 3 k_\vartheta^2) n_{[2]} + \lambda^2(\vartheta) k_\vartheta (-3 \vartheta^2 d_1^2 + k_\vartheta^2)(1 - n_{[2]}) \\
\sigma &= \{\lambda(\vartheta)[2 \vartheta^2 (\vartheta^2 - 1) d_1^2 - 2 \vartheta (1 + \vartheta^2) d_2 k_\vartheta] - \Lambda(\vartheta)[\vartheta (1 + \vartheta^2)(\vartheta d_1^2 - d_2 k_\vartheta)] + (1 - \vartheta^2)\} n_{[2]} \\
& \quad + \{\lambda(\vartheta)[2 \vartheta((1 + \vartheta^2) \vartheta d_2 + (\vartheta^2 - 1) k_\vartheta)] - \Lambda(\vartheta)[(1 + \vartheta^2)(d_2 (1 + \vartheta^2) + \vartheta k_\vartheta)]\}(1 - n_{[2]}) \\
\zeta &= \lambda^3(\vartheta) k_\vartheta \frac{5 \vartheta^4 d_1^4 + 4 \vartheta^6 d_1^4 + k_\vartheta^4 - 2 \vartheta^2 k_\vartheta^2 (5 d_1^2 + 2 k_\vartheta^2)}{2} n_{[2]} \\
& \quad + \lambda^3(\vartheta) \vartheta^2 \frac{- \vartheta^4 (3 + 4 \vartheta^2) d_1^2 + 2 \vartheta^2 (15 + 8 \vartheta^2) d_1^2 k_\vartheta^2 + 5 (-3 + 4 \vartheta^2) k_\vartheta^4}{2} (1 - n_{[2]}) \\
\tau &= \lambda^3(\vartheta) 3 \vartheta \frac{\vartheta^4 (1 + \vartheta^2) d_1^2 - 5 \vartheta^2 (2 + \vartheta^2) d_1^2 k_\vartheta^2 - 5 (-1 + \vartheta^2) d_1^2 k_\vartheta^4 + k_\vartheta^6}{2} n_{[2]} \\
& \quad + \lambda^3(\vartheta) k_\vartheta \frac{- 15 \vartheta^4 (1 + \vartheta^2) d_1^2 - 5 \vartheta^2 (-6 + \vartheta^2) d_1^2 k_\vartheta^2 + (-3 + 11 \vartheta^2) d_1^2 k_\vartheta^4 + k_\vartheta^6}{2} (1 - n_{[2]}) \\
v &= \left\{ - \lambda^2(\vartheta) 4 \vartheta^3 d_1^2 (\vartheta d_1^2 + d_2 k_\vartheta) + 2 \lambda(\vartheta) \vartheta (2 \vartheta d_1^2 + d_2 k_\vartheta) - \frac{1}{2} \right\} \frac{n_{[2]}}{d_1^2 + d_2^2} \\
& \quad + \left\{ \lambda^2(\vartheta) 4 \vartheta^3 d_1^2 (\vartheta d_1^2 - d_2 k_\vartheta) - 2 \lambda(\vartheta) \vartheta (2 \vartheta d_1^2 - d_2 k_\vartheta) + \frac{d_2}{2 d_1^2} \right\} \frac{1 - n_{[2]}}{d_1^2 + d_2^2} \\
\phi &= \{\lambda^2(\vartheta) 8 \vartheta^3 d_1^2 (\vartheta (-1 + \vartheta^2) d_1^2 - (1 + \vartheta^2) d_2 k_\vartheta) + \lambda(\vartheta) 4 \vartheta (2 \vartheta (1 - \vartheta^2) d_1^2 + (1 + \vartheta^2) d_2 k_\vartheta) \\
& \quad + (-1 - 6 \vartheta^2 + \vartheta^4) + \Lambda(\vartheta) \vartheta (1 + \vartheta^2)^2 (2 \vartheta d_1^2 - d_2 k_\vartheta)\} n_{[2]} \\
& \quad + \{\lambda^2(\vartheta) 8 \vartheta^3 d_1^2 (\vartheta (1 + \vartheta^2) d_2 + (1 + \vartheta^2) k_\vartheta) + \lambda(\vartheta) 4 \vartheta (k_\vartheta - \vartheta (2 (1 + \vartheta^2) d_2 + 3 d_2 k_\vartheta)) \\
& \quad + \Lambda(\vartheta) (1 + \vartheta^2)^2 (d_2 (1 + \vartheta^2) + d_2 k_\vartheta)\}(1 - n_{[2]})
\end{align*}
\]
Proposition

With assumptions (A1), (A7) it holds

\[
\int_{x_2}^{y_2-x_2} \frac{d^2}{d_1^2 + d_2^2} \frac{1}{r^5} \bigg|_{\vartheta=a}^{\vartheta=b} \, dd_2 = \lambda(\vartheta) \left[ \varphi \arctanh \frac{d_3}{r} + \chi \int_{\lambda}^{-3} (x, d_2, d_3) + \psi \frac{1}{r^3} \bigg|_{\vartheta=a}^{\vartheta=b} \right]_{d_2=-x_2}
\]

(A10)

where

\[
\varphi = \lambda^4(\vartheta) \{ -k_\vartheta(5\vartheta^4 d_1^4 - 10\vartheta^2 d_1^2 k_\vartheta^2 + k_\vartheta^4) n_{[2]} + \vartheta(\vartheta^4 d_1^4 - 10\vartheta^2 d_1^2 k_\vartheta^2 + 5k_\vartheta^4)(1 - n_{[2]}) \}
\]

\[
\chi = \lambda^4(\vartheta) \{ d_1^2(\vartheta^5 d_1^4 - 10\vartheta^3 d_1^2 k_\vartheta^2 + 5\vartheta^2 k_\vartheta^4) n_{[2]} + (5\vartheta^4 d_1^4 k_\vartheta - 10\vartheta^2 d_1^2 k_\vartheta^2 + k_\vartheta^4)(1 - n_{[2]}) \}
\]

\[
\psi = \lambda(\vartheta) \left\{ -\lambda^2(\vartheta)8\vartheta^3 d_1^2(\vartheta(-1 + \vartheta^2)d_1^2 - (1 + \vartheta^2)d_2 k_\vartheta) \\
+ \lambda(\vartheta)4\vartheta(\vartheta(-2 + 3\vartheta^2 + \vartheta^4)d_1^2 - (1 + \vartheta^2)^2 d_2 k_\vartheta) \\
- \lambda^2(\vartheta)2\vartheta(1 + \vartheta^2)^2(\vartheta d_1^2 - d_2 k_\vartheta)((1 + 2\vartheta^2)d_1^2 + 2k_\vartheta^2) \\
+ \lambda(\vartheta)k_\vartheta(3\vartheta(1 + 3\vartheta^2 + 2\vartheta^4)d_2 + (3 - 5\vartheta^2 - 2\vartheta^4)k_\vartheta - (-3 + 7\vartheta^2 + 18\vartheta^4 + 8\vartheta^6)d_1^2) \right\} n_{[2]}
\]

\[
\omega = \left\{ \lambda(\vartheta)2\vartheta(\vartheta(-1 + \vartheta^2)d_1^2 - (1 + \vartheta^2)d_2 k_\vartheta) \\
+ \lambda(\vartheta)k_\vartheta(\vartheta d_2 + (1 + \vartheta^2) + \vartheta k_\vartheta) - (-1 + \vartheta^2 + 2\vartheta^4)d_1^2 \right\} n_{[2]}
\]

\[
+ \{ \lambda(\vartheta)2\vartheta(\vartheta(1 + \vartheta^2)d_2 + (-1 + \vartheta^2)k_\vartheta) \\
- \lambda(\vartheta)(1 + \vartheta^2)(d_2(1 + \vartheta^2) + \vartheta k_\vartheta) \right\} (1 - n_{[2]})
\]
APPENDIX C: MATRICES FOR THE POTENTIAL KERNEL

C.1. Weakly singular kernel
Making reference to the notation of formulae (43)-(44), with assumptions (A7) and \( u_\varphi = 1 + \varphi^2 \), it holds
\[
\mathcal{L}^{uu} = u_\varphi^{-1/2} \left[ \mathcal{L}^A + \varphi^{-1} \mathcal{L}^C \right]
\]
where
\[
\mathcal{L}^A = \left[ k_\varphi, -\frac{\varphi}{2} d_1^2 \right], \quad \mathcal{L}^C = \left[ 0, -\frac{\varphi^2}{2} k_\varphi^2 \right]
\]
and
\[
\mathcal{A}^{uu} = \left[ d_2, \frac{1}{2} (d_1^2 + d_2^2) \right], \quad \mathcal{V}^{uu} = [-d_1^2, 0], \quad \mathcal{R}^{uu} = \left[ 0, \frac{k_\varphi}{2u_\varphi} \right];
\]

C.2. Strongly singular kernels
C.2.1. Kernel \( G_{up} \). Making reference to the notation of formulae (50)-(51) and assuming \( u_\varphi = 1 + \varphi^2 \), it holds
\[
\mathcal{L}^{up} = \left[ 0, -\frac{\varphi d_1}{\sqrt{u_\varphi}} \right], \quad \mathcal{A}^{up} = [0, d_1], \quad \mathcal{V}^{up} = [-d_1, 0], \quad \mathcal{R}^{up} = 0, \quad \mathcal{S}^{up} = 0
\]

C.2.2. Kernel \( G_{pu} \). Making reference to the notation of formulae (50)-(52) and assuming \( u_\varphi = 1 + \varphi^2 \), it holds
\[
\mathcal{L}^{pu} = u_\varphi^{-1/2} \left[ \mathcal{L}^A + \varphi^{-1} \mathcal{L}^C \right]
\]
where
\[
\mathcal{L}^A = [\partial n_2 - n_3, d_1 n_1 \varphi], \quad \mathcal{L}^C = [0, k_\varphi (\varphi n_3 + n_2)]
\]
and
\[
\mathcal{A}^{pu} = [-n_2, -d_1 n_1], \quad \mathcal{V}^{pu} = [d_1 n_1, -d_1^2 n_2], \quad \mathcal{R}^{pu} = \left[ 0, \frac{\varphi n_2 - n_3}{u_\varphi} \right], \quad \mathcal{S}^{pu} = 0
\]

C.3. Hyper singular kernel
Making reference to the notation of formulae (60)-(61), with assumptions (A1), (A7) and \( u_\varphi = 1 + \varphi^2 \), it holds
\[
\mathcal{L}^{pp} = \left[ 0, \frac{n_1 \varphi}{\sqrt{u_\varphi}} \right], \quad \mathcal{A}^{pp} = [0, -n_1], \quad \mathcal{V}^{pp} = [0, -d_1 n_2], \quad \mathcal{R}^{pp} = 0
\]
\[
\mathcal{S}^{pp} = \lambda(\varphi) \Lambda(\varphi) d_1 [n \cdot z, n \cdot \beta], \quad \mathcal{R}^{pp} = -\frac{\lambda(\varphi)}{(d_1^2 + d_2^2)} [n \cdot y, -d_1 n \cdot y]^{-1}
\]
where
\[
\alpha = \left[ 2\partial d_1 k_0^2 + u_\partial d_1(\partial d_1^2 + d_2 k_0), \partial u_\partial d_1^2 d_2 - k_0(d_1^2 + k_0^2), \frac{u_\partial d_2 + \partial k_0}{\lambda(\partial \cdot)} \right]
\]
\[
\beta = \left[ \partial u_\partial d_2 d_1^3 - k_0 d_1(d_1^2 + k_0^2), -2\partial^2 d_2 k_0 d_1^2 - (d_1^2 + k_0^2)(\partial d_1^2 + d_2 k_0), \frac{d_1^2 + k_0^2 + \partial d_2 k_0}{\lambda(\partial \cdot)} \right]
\]
\[
\gamma = \left[ \partial d_1^2 + d_2 k_0, d_1(\partial d_2 - k_0), 0 \right]
\]
\[
\gamma_1 = [-\gamma_2, \gamma_1, 0]
\]
\[
n = [n_1, n_2, n_3]
\]

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