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Hypersingular integral equation for multiple curved cracks problem in plane elasticity

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ABSTRACT

The complex variable function method is used to formulate the multiple curved crack problems into hypersingular integral equations. These hypersingular integral equations are solved numerically for the unknown function, which are later used to find the stress intensity factor, SIF, for the problem considered. Numerical examples for double circular arc cracks are presented.

1. Introduction

The crack geometry is one of the major factors that influences the resistance of many engineering structures. For two dimensional problems, the crack maybe straight or curve. Rice (1972) treated a straight crack subject to different load systems, and found that if a solution for the displacement field and SIF is known for any particular load system, then this information is sufficient to determine the SIF for other load system. Many researchers used perturbation technique when dealing with a slightly curved or kinked crack. Examples are Cotterell and Rice (1980), Martin (2000), and Panasyuk et al. (1977) for a various set of cracks positions. Kachanov (1985), based on superposition technique, solved a stress in elastic solid problems with many cracks. Besides many analytical works have been done, the needs for numerical technique for solving such a problem still remains. Examples are, works by Chen et al. (2003) and Chen, whom applied complex potential method to formulate the problems of multiple curved cracks into singular integral equations (Chen, 2004, 2007) and a curved crack into hypersingular integral equation (Chen, 1993, 2003). These equations were then solved numerically. Finite element method, which requires fine discretization in the vicinity of crack tips (Beghini et al., 1999a,b), and boundary element method, for degenerate scale problems (Chen et al., 2002), have also been used successfully, and the dual boundary integral representation, Hong and Chen (1988) and Chen and Hong (1999) gave a perspective view of the formulations of dual boundary element methods with emphasis on the regularization of hypersingular integrals and divergent series.

In this paper, a double circular arc cracks is formulated into hypersingular integral equation using a complex potential method. This is further described in Section 3. Then in Section 4, the equation is solved numerically using the so-called curved length coordinate method. In this method, we only need to collocate at $n+1$ points over a straight line, instead of a curve, which obviously needs fewer points, and hence provides faster convergence. Some numerical examples are presented in Section 6.

2. Finite part integral

Write

$$\oint_L \frac{G(t)\,dt}{(t-t_0)^2} = \int_{L\backslash l} \frac{G(t)\,dt}{(t-t_0)^2} + \oint_l \frac{G(t)\,dt}{(t-t_0)^2} \tag{1}$$

where $G(t)$ is a bounded function with continuous first and second derivatives, L is a smooth curve and $l=(t_0-\varepsilon, t_0+\varepsilon)$. The equal sign on integrals denote the hypersingular integral. The first integral on the right hand side of (1) is regular. For the second integral, we write

$$\oint_l \frac{G(t)\,dt}{(t-t_0)^2} = \int_l \frac{G(t)-G(t_0)-(t-t_0)G'(t_0)\,dt}{(t-t_0)^2} + G(t_0)\oint_l \frac{dt}{(t-t_0)^2} + G'(t_0)\oint_l \frac{dt}{(t-t_0)}. \tag{2}$$

Expand $G(t)$ around t_0 gives

$$I = \int_l \frac{G(t)-G(t_0)-(t-t_0)G'(t_0)\,dt}{(t-t_0)^2} = \frac{1}{2}G''(\eta)\int_l dt, \quad \eta \in (t_0, t).$$

Obviously, $I \to 0$ as $\varepsilon \to 0$. For the second integral (Beloserkovskii and Lifanov, 1985)

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$$\oint_{l} \frac{dt}{(t-t_0)^2} = -\frac{2}{\varepsilon}.$$

The third integral is of Cauchy type and along the straight line containing the singular point t_0, it vanishes. Therefore, the integral on the left side of (1) can be interpreted as a Hadamart finite part integral (Lifanov et al., 2004) as

$$\oint_{L} \frac{G(t)dt}{(t-t_0)^2} = \lim_{\varepsilon\to 0}\left(\int_{L\setminus l} \frac{G(t)dt}{(t-t_0)^2} - \frac{2G(t_0)}{\varepsilon}\right). \tag{3}$$

The finite part integral (3) is related to Cauchy principle value integral (Martin, 1992) by

$$\oint_{L} \frac{G(t)dt}{(t-t_0)^2} = -\frac{d}{dt}\int_{L} \frac{G(t)\,dt}{(t-t_0)}. \tag{4}$$

3. Hypersingular integral equation

The complex variable function method (Chen, 1993; Chen, 2004) is used to formulate the hypersingular integral equation for the multiple curved crack problems. Let $\phi'(z) = \Phi(z)$ and $\psi'(z) = \Psi(z)$ be two complex potentials. Then the stress $(\sigma_x, \sigma_y, \sigma_{xy})$, the resultant function (X, Y) and the displacement (u, v) are related to $\Phi(z)$ and $\Psi(z)$ (Muskhelishvili, 1953) as

$$\sigma_x + \sigma_y = 4Re\Phi(z)$$
$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{5}$$
$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} + c \tag{6}$$
$$2G(u+iv) = K\phi(z) - z\overline{\phi'(z)} - \psi(z) \tag{7}$$

where G is shear modulus of elasticity, $K = 3 - v$ for the plane strain problem, $K = (3-v)/(1+v)$ for the plane stress problem and v is Poisson's ratio and $z = x + iy$. The derivative in a specified direction (DISD) for Eq. (6) is defined as

$$J_1\left(z, \bar{z}, \frac{d\bar{z}}{dz}\right) = \frac{d}{dz}\{-Y + iX\} = \Phi(z) + \overline{\Phi(z)} + \frac{d\bar{z}}{dz}\left(z\overline{\Phi'(z)} + \overline{\Psi(z)}\right) = N + iT \tag{8}$$

where J_1 denotes the normal and tangential tractions along the segment $\overline{z, z+dt}$. Note that the value of J_1 not only depends on the position of point z, but also on the direction of the segment $d\bar{z}/dz$.

The complex potential in plane elasticity is obtained by placing two point dislocations with intensity $H(-H)$ at point $z = t(z = t + dt)$, yield

$$\phi(z) = -H\frac{dt}{z-t}, \qquad \psi(z) = -\overline{H}\frac{dt}{t-z} - H\frac{d\bar{t}}{t-z} + H\frac{\bar{t}dt}{(t-z)^2}. \tag{9}$$

Making substitution of $H(\overline{H})$ by $-g(t)/2\pi(-\overline{g(t)}/2\pi)$ in (9) and performing integration on the right side of (9), gives

$$\phi(z) = \frac{1}{2\pi}\int_L \frac{g(t)dt}{t-z}$$
$$\psi(z) = \frac{1}{2\pi}\int_L \frac{g(t)d\bar{t}}{t-z} + \frac{1}{2\pi}\int_L \frac{\overline{g(t)}dt}{t-z} - \frac{1}{2\pi}\int_L \frac{\bar{t}g(t)dt}{(t-z)^2} \tag{10}$$

where L denotes the curve crack.

Substituting (10) into (7), and letting z approaches t_0^+ and t_0^-, which are located on the upper and lower sides of the crack faces, then using the generalized Plemelj equation, and rewriting t_0 as t, the following results is obtained (Chen, 1993)

$$2G(u(t) + iv(t))_j = i(\kappa+1)g(t) \qquad (t \in L) \tag{11}$$

where $(u(t) + iv(t))_j = (u(t) + iv(t))^+ - (u(t) + iv(t))^-$ denotes the cracks opening displacement (COD) for the curved cracks. It is well known that the COD possesses the following properties:

$$g(t) = O[(t - t_{A_1})^{1/2}] \text{ at the vicinity of the left crack tip } A_1$$
$$g(t) = O[(t - t_{A_2})^{1/2}] \text{ at the vicinity of the right crack tip } A_2. \tag{12}$$

All of the above fundamental works are scattered everywhere, for example see Chen (1993).

Chen (1993) obtained the hypersingular integral equation for a single curved crack problem by placing two point dislocations at the points $z = t$ and $z = t + dt$, and is given by

$$\frac{1}{\pi}\oint_{L} \frac{g(t)\,dt}{(t-t_0)^2} + \frac{1}{2\pi}\int_L K_1(t,t_0)g(t)\,dt + \frac{1}{2\pi}\int_L K_2(t,t_0)\overline{g(t)}\,dt$$
$$= N(t_0) + iT(t_0), \quad t_0 \in L \tag{13}$$

where

$$K_1(t,t_0) = -\frac{1}{(t-t_0)^2} + \frac{1}{(\bar{t}-\bar{t}_0)^2}\frac{d\bar{t}_0}{dt_0}\frac{d\bar{t}}{dt}$$
$$K_2(t,t_0) = -\frac{1}{(\bar{t}-\bar{t}_0)^2}\left(\frac{d\bar{t}}{dt} + \frac{d\bar{t}_0}{dt_0}\right) - \frac{2(t-t_0)}{(\bar{t}-\bar{t}_0)^3}\frac{d\bar{t}_0}{dt_0}\frac{d\bar{t}}{dt} \tag{14}$$

and $g(t)$ is the dislocation distribution along the crack curves. In (13), the first integral with equal sign on it denotes the hypersingular integral, and should be defined in the sense of Hadamart finite part integral.

Now consider the two curved cracks problem (see Fig. 1a). For the crack-1, $(r = r_1)$, if the point dislocation is placed at points $z = t_{10}$ and $dz = dt_{10}$, and $g_1(t_1)$ is the dislocation doublet distribution along the crack-1, and the traction is applied on t_{10} then the hypersingular integral equation for crack-1 is

$$\frac{1}{\pi}\oint_{L_1} \frac{g_1(t_1)\,dt_1}{(t_1-t_{10})^2} + \frac{1}{2\pi}\int_{L_1} K_1(t_1,t_{10})g_1(t_1)\,dt_1$$
$$+ \frac{1}{2\pi}\int_{L_1} K_2(t_1,t_{10})\overline{g_1(t_1)}\,dt_1 = N_{11}(t_{10}) + iT_{11}(t_{10}), \quad t_{10} \in L_1 \tag{15}$$

where $N_{11}(t_{10}) + iT_{11}(t_{10})$ denotes the traction influence on crack-1 caused by dislocation doublet distribution, $g_1(t_1)$, on crack-1, and

$$K_1(t_1,t_{10}) = -\frac{1}{(t_1-t_{10})^2} + \frac{1}{(\bar{t}_1-\bar{t}_{10})^2}\frac{d\bar{t}_{10}}{dt_{10}}\frac{d\bar{t}_1}{dt_1}$$
$$K_2(t_1,t_{10}) = -\frac{1}{(\bar{t}_1-\bar{t}_{10})^2}\left(\frac{d\bar{t}_1}{dt_1} + \frac{d\bar{t}_{10}}{dt_{10}}\right) - \frac{2(t_1-t_{10})}{(\bar{t}_1-\bar{t}_{10})^3}\frac{d\bar{t}_{10}}{dt_{10}}\frac{d\bar{t}_1}{dt_1}.$$

The influence from the dislocation doublet distribution on crack-2, $(r = r_2)$, gives

$$\frac{1}{\pi}\int_{L_2} \frac{g_2(t_2)\,dt_2}{(t_2-t_{10})^2} + \frac{1}{2\pi}\int_{L_2} K_1(t_2,t_{10})g_2(t_2)\,dt_2$$
$$+ \frac{1}{2\pi}\int_{L_2} K_2(t_2,t_{10})\overline{g_2(t_2)}\,dt_2 = N_{12}(t_{10}) + iT_{12}(t_{10}) \tag{16}$$

where $N_{12}(t_{10}) + iT_{12}(t_{10})$ is the traction influence on crack-1 caused by dislocation doublet distribution, $g_2(t_2)$, on crack-2, and

$$K_1(t_2,t_{10}) = -\frac{1}{(t_2-t_{10})^2} + \frac{1}{(\bar{t}_2-\bar{t}_{10})^2}\frac{d\bar{t}_{10}}{dt_{10}}\frac{d\bar{t}_2}{dt_2}$$
$$K_2(t_2,t_{10}) = -\frac{1}{(\bar{t}_2-\bar{t}_{10})^2}\left(\frac{d\bar{t}_2}{dt_2} + \frac{d\bar{t}_{10}}{dt_{10}}\right) - \frac{2(t_2-t_{10})}{(\bar{t}_2-\bar{t}_{10})^3}\frac{d\bar{t}_{10}}{dt_{10}}\frac{d\bar{t}_2}{dt_2}.$$

Note that since $t_2 - t_{10} \neq 0$, all three integrals in (16) are regular, and that $g_1(t)$ and $g_2(t)$ satisfy (12).

By superposition of the dislocation doublet distribution $g_1(t_1)$ along the curved crack-1 (15) and the dislocation doublet distribution $g_2(t_2)$ along the curved crack-2 (16), we obtain the hypersingular integral equation for crack-1 which is

$$\frac{1}{\pi}\oint_{L_1} \frac{g_1(t_1)\,dt_1}{(t_1-t_{10})^2} + \frac{1}{2\pi}\int_{L_1} K_1(t_1,t_{10})g_1(t_1)\,dt_1$$
$$+ \frac{1}{2\pi}\int_{L_1} K_2(t_1,t_{10})\overline{g_1(t_1)}\,dt_1 + \frac{1}{\pi}\int_{L_2} \frac{g_2(t_2)\,dt_2}{(t_2-t_{10})^2}$$
$$+ \frac{1}{2\pi}\int_{L_2} K_1(t_2,t_{10})g_2(t_2)\,dt_2 + \frac{1}{2\pi}\int_{L_2} K_2(t_2,t_{10})\overline{g_2(t_2)}\,dt_2$$
$$= N_1(t_{10}) + iT_1(t_{10}), \tag{17}$$

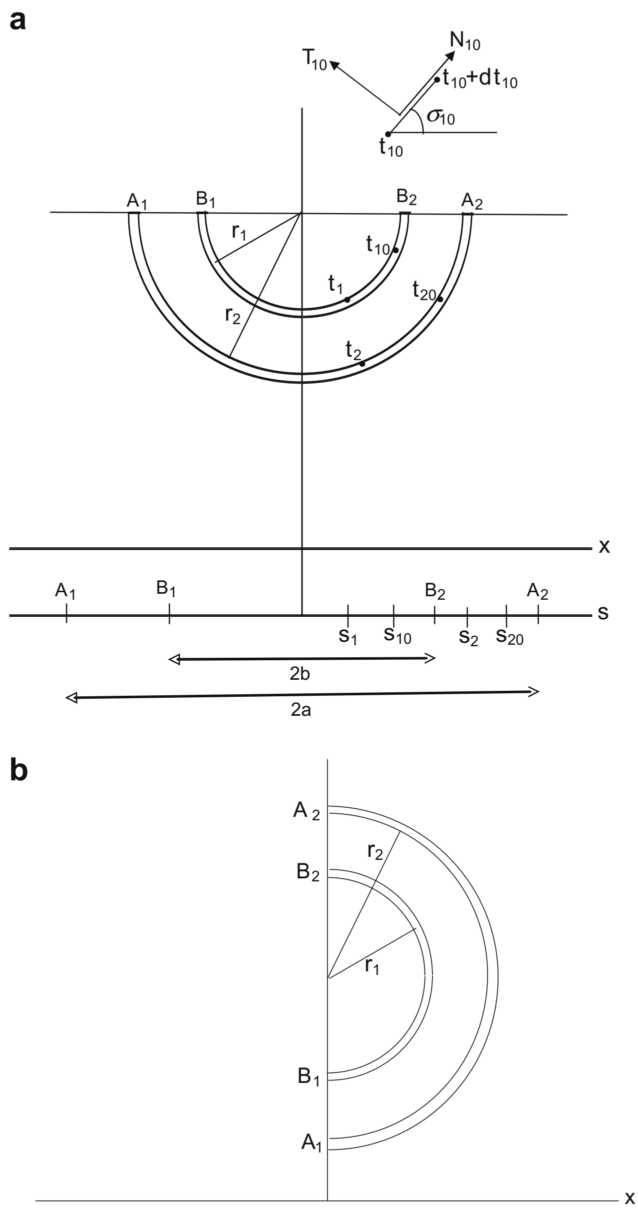

Fig. 1. double circular arc cracks (a) double circular arcs, (b) double circular arcs with 90° rotation.

where

$$N_1(t_{10})+iT_1(t_{10})=N_{11}(t_{10})+iT_{11}(t_{10})+N_{12}(t_{10})+iT_{12}(t_{10})\quad (t_{10}\in L_1)$$

is the traction applied at point t_{10} of crack-1, which is derived from the boundary condition.

The first three integrals in (17) represent the effect on crack-1 caused by the dislocations on the crack-1 itself, whereas the second three integrals represent the effect of the dislocations on crack-2.

Similarly, the hypersingular integral equation for crack-2 is

$$\begin{aligned}&\frac{1}{\pi}\oint_{L_2}\frac{g_2(t_2)dt_2}{(t_2-t_{20})^2}+\frac{1}{2\pi}\int_{L_2}K_1(t_2,t_{20})g_2(t_2)dt_2+\frac{1}{2\pi}\\ &\times\int_{L_2}K_2(t_2,t_{20})\overline{g_2(t_2)}dt_2+\frac{1}{\pi}\int_{L_1}\frac{g_1(t_1)dt_1}{(t_1-t_{20})^2}+\frac{1}{2\pi}\\ &\times\int_{L_1}K_1(t_1,t_{20})g_1(t_1)dt_1+\frac{1}{2\pi}\int_{L_1}K_2(t_1,t_{20})\overline{g_1(t_1)}dt_1\\ &=N_2(t_{20})+iT_2(t_{20})\end{aligned}\tag{18}$$

where

$$N_2(t_{20})+iT_2(t_{20})=N_{21}(t_{20})+iT_{21}(t_{20})+N_{22}(t_{20})+iT_{22}(t_{20})\quad (t_{20}\in L_2)$$

is the traction applied at point t_{20} of crack-2, and

$$\begin{aligned}K_1(t_2,t_{20})&=-\frac{1}{(t_2-t_{20})^2}+\frac{1}{(\bar{t}_2-\bar{t}_{20})^2}\frac{d\bar{t}_{20}}{dt_{20}}\frac{d\bar{t}_2}{dt_2}\\ K_2(t_2,t_{20})&=-\frac{1}{(\bar{t}_2-\bar{t}_{20})^2}\left(\frac{d\bar{t}_2}{dt_2}+\frac{d\bar{t}_{20}}{dt_{20}}\right)-\frac{2(t_2-t_{20})}{(\bar{t}_2-\bar{t}_{20})^3}\frac{d\bar{t}_{20}}{dt_{20}}\frac{d\bar{t}_2}{dt_2}\\ K_1(t_1,t_{20})&=-\frac{1}{(t_1-t_{20})^2}+\frac{1}{(\bar{t}_1-\bar{t}_{20})^2}\frac{d\bar{t}_{20}}{dt_{20}}\frac{d\bar{t}_1}{dt_1}\\ K_2(t_1,t_{20})&=-\frac{1}{(\bar{t}_1-\bar{t}_{20})^2}\left(\frac{d\bar{t}_1}{dt_1}+\frac{d\bar{t}_{20}}{dt_{20}}\right)-\frac{2(t_1-t_{20})}{(\bar{t}_1-\bar{t}_{20})^3}\frac{d\bar{t}_{20}}{dt_{20}}\frac{d\bar{t}_1}{dt_1}.\end{aligned}$$

In (18), the first three integrals represent the effect on crack-2 caused by the dislocation on crack-2 itself, and the second three integrals represent the effect of the dislocation on crack-1. Eqs. (17) and (18) are to be solved for $g_1(t_1)$ and $g_2(t_2)$.

It is obvious that if the two cracks are far apart, we have $|t_2-t_{10}|$ and $|t_1-t_{20}|$ approach infinity. These lead to the second three integrals in Eqs. (17) and (18) vanish. Then the solutions for Eqs. (17) and (18) approach the solution for a single crack problem, and a closed form solution is available (Cotterell and Rice, 1980).

4. Curved length coordinate methods

We map the two curve cracks configurations on a real axis s with an interval of $2a$ and $2b$, respectively. The mapping is expressed by the functions $t_1(s_1)$ and $t_2(s_2)$ as follows

$$g_1(t_1)|_{t_1=t_1(s_1)}=\sqrt{a^2-s_1^2}H_1(s_1)\ \text{ where }\ H_1(s_1)=H_{11}(s_1)+iH_{12}(s_1)\tag{19}$$

and

$$g_2(t_2)|_{t_2=t_2(s_2)}=\sqrt{b^2-s_2^2}H_2(s_2)\ \text{ where }\ H_2(s_2)=H_{21}(s_2)+iH_{22}(s_2).\tag{20}$$

In Eqs. (19) and (20), $g_1(t_1)$ and $g_2(t_2)$ are chosen in such a way that they must satisfy the behavior of the COD at the vicinity of crack tips (see Eq. (12)). Using these transformations, all integrals in (17) and (18) can be transformed into integrals on the real axis s.

Substitution Eqs. (19) and (20) into (17) and (18), respectively, yield

$$\begin{aligned}&I_1(s_{10})+I_2(s_{10})+I_3(s_{10})+I_4(s_{10})+I_5(s_{10})+I_6(s_{10})\\ &\quad=N_1(s_{10})+iT_1(s_{10})\end{aligned}\tag{21}$$

and

$$\begin{aligned}&L_1(s_{20})+L_2(s_{20})+L_3(s_{20})+L_4(s_{20})+L_5(s_{20})+L_6(s_{20})\\ &\quad=N_2(s_{20})+iT_2(s_{20})\end{aligned}\tag{22}$$

where

$$\begin{aligned}I_1(s_{10})&=\frac{1}{\pi}\oint_{-a}^{a}\sqrt{a^2-s_1^2}\,H_1(s_1)\frac{A_1(s_1,s_{10})ds_1}{(s_1-s_{10})^2}\\ I_2(s_{10})&=\frac{1}{2\pi}\int_{-a}^{a}\sqrt{a^2-s_1^2}\,H_1(s_1)B_1(s_1,s_{10})ds_1\\ I_3(s_{10})&=\frac{1}{2\pi}\int_{-a}^{a}\sqrt{a^2-s_1^2}\,\overline{H_1(s_1)}C_1(s_1,s_{10})ds_1\\ I_4(s_{10})&=\frac{1}{\pi}\int_{-b}^{b}\sqrt{b^2-s_2^2}\,H_2(s_2)\frac{D_1(s_2,s_{10})ds_2}{(s_2-s_{10})^2}\\ I_5(s_{10})&=\frac{1}{2\pi}\int_{-b}^{b}\sqrt{b^2-s_2^2}\,H_2(s_2)E_1(s_2,s_{10})ds_2\\ I_6(s_{10})&=\frac{1}{2\pi}\int_{-b}^{b}\sqrt{b^2-s_2^2}\,\overline{H_2(s_2)}F_1(s_2,s_{10})ds_2\end{aligned}$$

$$L_1(s_{20}) = \frac{1}{\pi} \fint_{-b}^{b} \sqrt{b^2 - s_2^2}\, H_2(s_2) \frac{A_2(s_2, s_{20}) ds_2}{(s_2 - s_{20})^2}$$

$$L_2(s_{20}) = \frac{1}{2\pi} \int_{-b}^{b} \sqrt{b^2 - s_2^2}\, H_2(s_2) B_2(s_2, s_{20}) ds_2$$

$$L_3(s_{20}) = \frac{1}{2\pi} \int_{-b}^{b} \sqrt{b^2 - s_2^2}\, \overline{H_2(s_2)} C_2(s_2, s_{20}) ds_2$$

$$L_4(s_{20}) = \frac{1}{\pi} \int_{-a}^{a} \sqrt{a^2 - s_1^2}\, H_1(s_1) \frac{D_2(s_1, s_{20}) ds_1}{(s_1 - s_{20})^2}$$

$$L_5(s_{20}) = \frac{1}{2\pi} \int_{-a}^{a} \sqrt{a^2 - s_1^2}\, H_1(s_1) E_2(s_1, s_{20}) ds_1$$

$$L_6(s_{20}) = \frac{1}{2\pi} \int_{-a}^{a} \sqrt{a^2 - s_1^2}\, \overline{H_1(s_1)} F_2(s_1, s_{20}) ds_1$$

and, $A_1, B_1, C_1, D_1, E_1, F_1$ and $A_2, B_2, C_2, D_2, E_2, F_2$ are, respectively, given by

$$A_1(s_1, s_{10}) = \frac{(s_1 - s_{10})^2}{(t_1 - t_{10})^2} \frac{dt_1}{ds_1}; \quad B_1(s_1, s_{10}) = K_1(t_1, t_{10}) \frac{dt_1}{ds_1};$$

$$C_1(s_1, s_{10}) = K_2(t_1, t_{10}) \frac{dt_1}{ds_1}; \quad D_1(s_2, s_{10}) = \frac{(s_2 - s_{10})^2}{(t_2 - t_{10})^2} \frac{dt_2}{ds_2};$$

$$E_1(s_2, s_{10}) = K_1(t_2, t_{10}) \frac{dt_2}{ds_2}; \quad F_1(s_2, s_{10}) = K_2(t_2, t_{10}) \frac{dt_2}{ds_2},$$

and

$$A_2(s_2, s_{20}) = \frac{(s_2 - s_{20})^2}{(t_2 - t_{20})^2} \frac{dt_2}{ds_2}; \quad B_2(s_2, s_{20}) = K_1(t_2, t_{20}) \frac{dt_2}{ds_2};$$

$$C_2(s_2, s_{20}) = K_2(t_2, t_{20}) \frac{dt_2}{ds_2}; \quad D_2(s_1, s_{20}) = \frac{(s_1 - s_{20})^2}{(t_1 - t_{20})^2} \frac{dt_1}{ds_1};$$

$$E_2(s_1, s_{20}) = K_1(t_1, t_{20}) \frac{dt_1}{ds_1}; \quad F_2(s_1, s_{20}) = K_2(t_1, t_{20}) \frac{dt_1}{ds_1}.$$

5. Quadrature rules

In solving Eq. (21) and (22), we used the following integration rules (Mayrhofer and Fischer, 1992), for the hypersingular and regular integrals, respectively,

$$\frac{1}{\pi} \fint_{-a}^{a} \frac{\sqrt{a^2 - s^2}\, G(s)\, ds}{(s - s_0)^2} = \sum_{j=1}^{M+1} W_j(s_0) G(s_j) \quad (|s_0| < a) \tag{23}$$

and

$$\frac{1}{\pi} \int_{-a}^{a} \sqrt{a^2 - s^2}\, G(s)\, ds = \frac{1}{M+2} \sum_{j=1}^{M+1} (a^2 - s_j^2) G(s_j), \tag{24}$$

where $G(s)$ is a given regular function, $M \in \mathbb{Z}$,

$$s_j = a \cos\left(\frac{j\pi}{M+2}\right), \quad j = 1, 2, \ldots, M+1,$$

and

$$W_j(s_0) = -\frac{2}{M+2} \sum_{n=0}^{M} (n+1) \sin\left(\frac{j\pi}{M+2}\right) \sin\left(\frac{(n+1)j\pi}{M+2}\right) U_n\left(\frac{s_0}{a}\right),$$

and the observation (singular) points

$$s_0 = s_{0,k} = a \cos\left(\frac{k\pi}{M+2}\right), \quad k = 1, 2, \ldots, M+1.$$

Here $U_n(t)$ is a Chebyshev polynomial of the second kind, defined by

$$U_n(t) = \frac{\sin((n+1)\theta)}{\sin\theta}, \quad t = \cos\theta.$$

Denote

$$V_j^n = \sin\left(\frac{j\pi}{M+2}\right) \sin\left(\frac{(n+1)j\pi}{M+2}\right),$$

$H_1(s)$ and $H_2(s)$ are evaluated using

$$H_1(s) = \sum_{n=0}^{M} c_{1n} U_n\left(\frac{s}{a}\right), \quad |s| \leqslant a \tag{25}$$

and

$$H_2(s) = \sum_{n=0}^{M} c_{2n} U_n\left(\frac{s}{b}\right), \quad |s| \leqslant b \tag{26}$$

where

$$c_{1n} = \frac{2}{M+2} \sum_{j=1}^{M+1} V_j^n H_1(s_1), \tag{27}$$

$$c_{2n} = \frac{2}{M+2} \sum_{j=1}^{M+1} V_j^n H_2(s_2) \tag{28}$$

and $H_1(s_1)$ and $H_2(s_2)$ are defined in (19) and (20), respectively.

6. Numerical examples

One of the most interest quantities in solving the crack problem is the SIF. The SIF at the left tip of the inner and outer cracks are, respectively, defined as

$$K_{B_j} = (K_1 - iK_2)_{B_j} = \sqrt{2\pi} \lim_{t \to t_{B_j}} \sqrt{|t - t_{B_j}|}\, g_1'(t) \tag{29}$$

and

$$K_{A_j} = (K_1 - iK_2)_{A_j} = \sqrt{2\pi} \lim_{t \to t_{A_j}} \sqrt{|t - t_{A_j}|}\, g_2'(t), \quad j = 1, 2. \tag{30}$$

where $g_1'(t)$ and $g_2'(t)$ are obtained by solving Eqs. (21) and (22) simultaneously.

As described earlier, when the two cracks are far apart, the equations for a double circular arc cracks become an equation for a single circular arc, and the exact solution of the SIF for a single circular arc crack with the remote traction case of $\sigma_x^\infty, \sigma_y^\infty$ and σ_{xy}^∞ is given by (Cotterell and Rice, 1980)

$$\begin{aligned} K_1(\pi a)^{1/2} \Bigg\{ & \left[\left(\frac{\sigma_y^\infty + \sigma_x^\infty}{2} \right) - \left(\frac{\sigma_y^\infty - \sigma_x^\infty}{2} \right) \sin^2(\alpha/2) \cos^2(\alpha/2) \right] \\ & \times \frac{\cos(\alpha/2)}{[1 + \sin^2(\alpha/2)]} \left(\frac{\sigma_y^\infty - \sigma_x^\infty}{2} \right) \cos(3\alpha/2) \\ & - \sigma_{xy}^\infty \left[\sin(3\alpha/2) + \sin^3(\alpha/2) \right] \Bigg\} \end{aligned} \tag{31}$$

$$\begin{aligned} K_2 = (\pi a)^{1/2} \Bigg\{ & \left[\left(\frac{\sigma_y^\infty + \sigma_x^\infty}{2} \right) - \left(\frac{\sigma_y^\infty - \sigma_x^\infty}{2} \right) \sin^2(\alpha/2) \cos^2(\alpha/2) \right] \\ & \times \frac{\sin(\alpha/2)}{[1 + \sin^2(\alpha/2)]} \left(\frac{\sigma_y^\infty - \sigma_x^\infty}{2} \right) \sin(3\alpha/2) \\ & + \sigma_{xy}^\infty \left[\cos(3\alpha/2) + \cos(\alpha/2) \sin^2(\alpha/2) \right] \Bigg\}. \end{aligned} \tag{32}$$

Table 1
The SIF for single circular arc crack: a comparison between exact and curve length technique.

σ	K_{1B1}	K_{1B1}(exact)	K_{2B1}	K_{2B1}(exact)
10	0.9886	0.9887	0.0865	0.0865
20	0.9562	0.9560	0.1687	0.1686
30	0.9054	0.9053	0.2426	0.2426
40	0.8417	0.8413	0.3064	0.3062
50	0.7695	0.7690	0.3590	0.3586
60	0.6934	0.6928	0.4007	0.4000
70	0.6168	0.6164	0.4320	0.4316
80	0.5424	0.5421	0.4555	0.4549
90	0.4716	0.4714	0.4721	0.4714

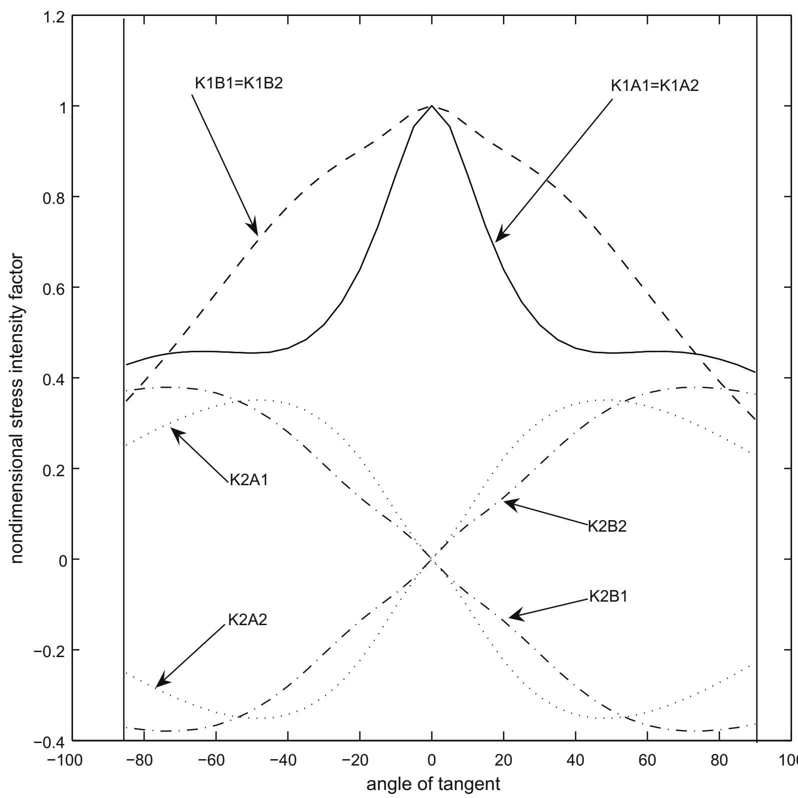

Fig. 2. The SIF for the cracks when σ is changing, Fig. 1a.

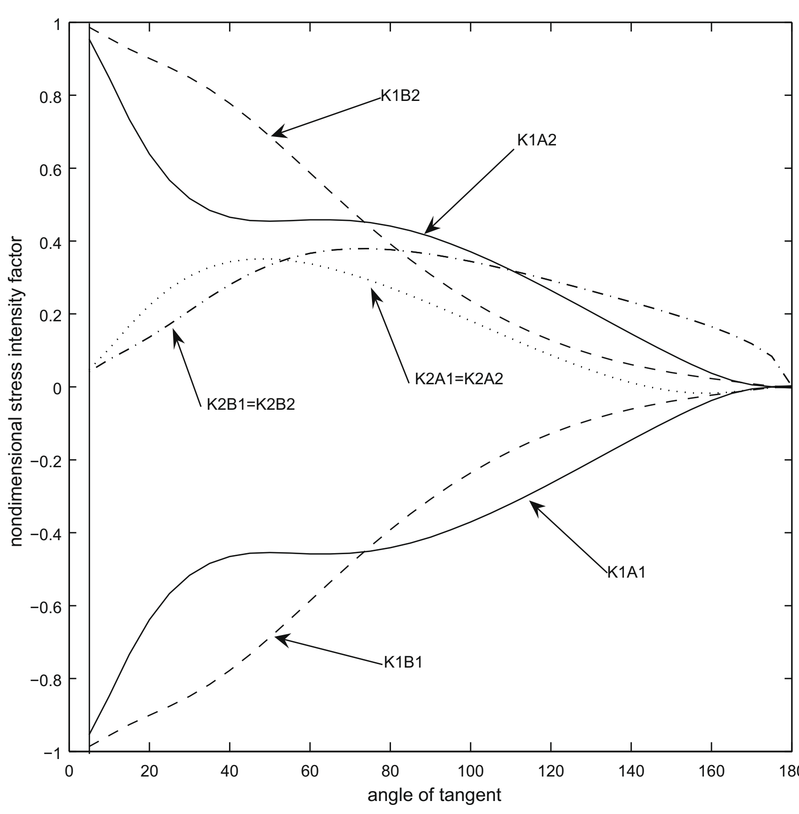

Fig. 3. The SIF for the cracks having 90° rotation, Fig. 1b.

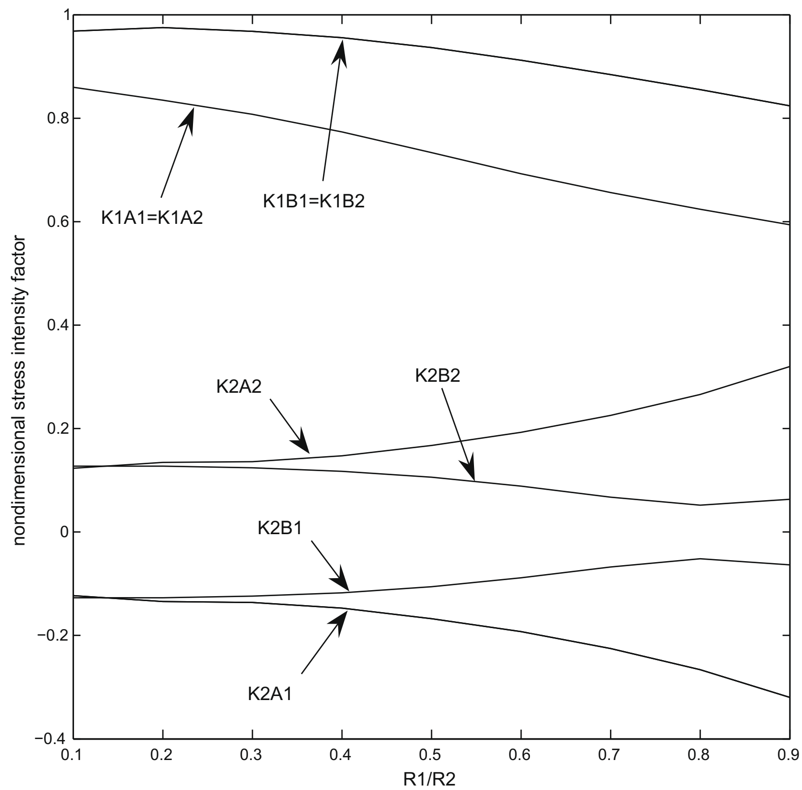

Fig. 4. The SIF for the cracks when r_1/r_2 is changing, Fig. 1a.

For the crack subjected to the remote tension $\sigma_x^\infty = \sigma_y^\infty = p = 1$, Eqs. (31) and (32) lead to

$$K_1 = (\pi a)^{1/2} \frac{\cos(\alpha/2)}{[1+\sin^2(\alpha/2)]} \tag{33}$$

$$K_2 = (\pi a)^{1/2} \frac{\sin(\alpha/2)}{[1+\sin^2(\alpha/2)]}. \tag{34}$$

This exact solution of a single circular arc crack (Eqs. (33) and (34)) provides us a simple check on the accuracy of the technique used. We compare our results for a single cracks with the exact solution numerically, and the figures are shown in Table 1. It can be seen from Table 1 that the maximum error is not more than 0.17%.

Now, for the double circular arc cracks (Fig. 1a), Fig. 2 shows the nondimensional SIF is plotted against σ, the position angle at the cracks tips. It is found that the real parts of SIF at the left crack tips A_1 and B_1 are equal to those of right crack tips A_2 and B_2 ($K_{1A1} = K_{1A2}$ and $K_{1B1} = K_{1B2}$), whereas the imaginary parts are of opposite sign ($K_{2A2} = -K_{2A1}$ and $K_{2B2} = -K_{2B1}$). These observations are in agreement with those of Chen (2003) for a single crack case. As the cracks are rotated 90^0 counter clockwise (Fig. 1b), Fig. 3 shows that the real parts for both cracks are of opposite sign ($K_{1A2} = -K_{1A1}$ and $K_{1B2} = -K_{1B1}$) whereas the imaginary parts are equals ($K_{2A1} = K_{2A2}$ and $K_{2B1} = K_{2B2}$). It is worth to note that by observing Figs. 2 and 3 for σ is between 0 and $\pi/2$, as expected, the values of SIF at the right cracks tips of Fig. 1a are equal to those of lower cracks tips of Fig. 1b, except for $K_{1A1}, K_{1B1}, K_{2A1}$ and K_{2B1} which are negative of $K_{1A2}, K_{1B2}, K_{2A1}$ and K_{2B1}, respectively. These behavior of SIF validate our results.

The effect of the distance between two cracks is also investigated, and the results are shown in Fig. 4, where the SIF are evaluated at $\sigma = 15^0$. Here we used $r = r_1/r_2$. Note that K_{1A1} and K_{2A1} denote mode I and mode II of the SIF at the crack tip A_1.

7. Conclusion

In the present work, the hypersingular integral equation for the double circular arc cracks has been formulated. The curved length coordinate method is used to obtain numerical solutions to the hypersingular integral equation. In this method, the double curved cracks is projected into a straight line, which require less collocation points, hence give faster convergence. It also allows us to evaluate the SIF for a deep curved crack configuration, as in our examples. The SIF for the mentioned crack subject to rotation are presented. To the best of authors knowledge the above numerical results for the double circular arc cracks configuration are new.

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