

# A nonequilibrium irreversible thermodynamics model for material damping

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## Abstract

Material damping in a continuum, which inherently involves multiple coupled irreversible thermodynamic processes, is associated with irreversible interchanges of various forms of energy. In this paper, a hybrid framework of internal state variables and extended state variables is proposed to formulate a material damping model for complex materials. For a typical simple material, damping model is developed in the framework of extended irreversible thermodynamics. Nonequilibrium quantities, i.e. thermodynamic fluxes, are introduced as extended state variables to supplement for the description of a local nonequilibrium state. The relaxation characteristics of these thermodynamic fluxes, which are a symbol of nonequilibrium characteristics, are modeled by first-order relaxation equations. For a typical system, we introduced heat flux and nonequilibrium (viscous) stresses as thermodynamic fluxes corresponding to irreversible heat transport and viscous processes. Coupling between equilibrium and nonequilibrium mechanical and thermal fields are modeled. Explicit expressions for specific free energy, specific entropy, and specific internal energy are derived. The energy balance equation that governs interchanges of various forms of energy is then obtained to complete the formulation. We have investigated various limiting cases of the developed model and related these limiting cases to typical thermodynamic damping models. We have applied the model to study the dissipation characteristics of longitudinal vibrations of a rod and then compared with other damping models.

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## 1. Introduction

Even though, we understand the elastic behavior of a structure, reasonably accurately, we do not understand the damping in the structure to the same level. The damping is a very desirable structural characteristic to attenuate vibrations, increase the fatigue life of a structure, and eliminate the undesired noise. The damping of a structure is primarily due to the material damping and possible dissipation effects by joints, fasteners and

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interfaces. To date, it is very difficult to accurately estimate the damping in a structure and therefore estimate the associated small changes in natural frequencies and other dynamic characteristics of the structure. However, these small changes are very important in many applications such as health monitoring of structural systems and estimating the residual strength of a structure. A precise material damping model of highly physical insight will lead to structural dynamic models of improved accuracy for simulation, system diagnostics and identifications, and design of active/passive control systems.

### 1.1. Mechanisms of material damping

Let us consider an elastic cantilever beam that is isolated in vacuum. When it is released from an initially deflected position, the beam vibrates with gradually decreasing amplitudes and finally comes to rest at an unstrained position. This simple case manifests the characteristics of material damping. Here, mechanical energy is converted into another form of energy, i.e., the heat, which is stored in the closed system, through a thermodynamic process, i.e., the thermoelastic process. In the thermoelastic process, it involves a coupling between the equilibrium thermal and mechanical fields. For a material of viscous characteristics, rather than the thermoelastic damping, viscous process and its coupling with other irreversible processes attribute to dissipation and damping. For a more general material, multiple irreversible processes at disparate scales may also contribute to the total dissipation. Thus, a precise damping model needs to describe various irreversible processes that can be at disparate scales, their inter-field coupling, and conversion of various forms of energy. This has to be developed in the framework of irreversible thermodynamics.

### 1.2. Framework of irreversible thermodynamics

Before we start formulating our damping model, we give a brief discussion on how we justify our thermodynamics framework from various irreversible thermodynamics theories. Starting from 1930s, Onsager, Prigogine, Eckart, Mexiner, de Groot, Mazur and others succeeded in formulating the principles of irreversible thermodynamics by making some restrictive assumptions (Onsager, 1931a,b; Onsager and Machlup, 1953). The concept of “state” is extended from a global description of a continuum in thermostatics (Buchdahl, 1966) to a local description of material point in the continuum. In other words, every material point that constructs the continuum is assumed approximately close to a local thermodynamic equilibrium state at any given instant (Fung, 1965; Prigogine, 1955; Jou et al., 1996). Therefore, state variables like temperature that are well defined in thermostatics for a global system can be used to specify a local equilibrium state at a material point. Today, this theory is known by the name *classical irreversible thermodynamics* (CIT). Besides the classical set of state variables, Onsager et al. introduced thermodynamic fluxes to describe irreversible processes. At any instant, thermodynamic fluxes instantaneously respond to the generalized forces by a linear relation that holds the Onsager–Casimir reciprocal principle (Onsager, 1931a,b; Casimir, 1945). These thermodynamic fluxes are not considered as additional state variables; therefore, do not appear in state functions like internal energy, free energy, etc., as defined in thermostatics. In 1943, Bridgeman addressed a balance law of entropy (i.e., the evolution of entropy): “(net entropy leaving a close region) = (entropy created within the region) – (increase of entropy localized in this region)”. In other word, the rate of change of entropy within a region is contributed by entropy flux through the region and entropy production created inside the region. In CIT, entropy flux only depends on heat flux, which is the thermodynamic flux for irreversible heat transfer process. The rate of entropy production is in a bilinear form of fluxes and their associated generalized forces. The nonnegativity of rate of entropy production, i.e. the dissipation inequality or Clausius–Duhem inequality, grants the irreversibility of dissipative processes and states the second law of CIT.

Rational thermodynamics (RT) was initially introduced by Truesdell, Coleman, Noll and others to enlarge the scope of the application of nonequilibrium thermodynamics beyond CIT (Truesdell, 1984; Coleman and Gurtin, 1967; Day, 1972; Jou et al., 1996). In RT, materials are assumed having memory. This means, at any given instant, dependent variables cannot be determined by only instantaneous values of independent variables, but by their entire histories. In some ways, this modifies the concept of state in CIT, where a state is well defined and determined by a set of measurable variables at only current instant. In RT, complementary variables (like internal energy, heat flux, stress tensor and entropy) are related to the entire history of indepen-

dent variables (like deformation and temperature) by functional relationships. In 1940s, Bridgeman proposed an introduction of “a large scale thermodynamic parameter of state”, to reduce the RT representation (Bridgeman, 1943; Kestin, 1992) and to avoid the use of functional equations. These state parameters, which extend the state space of CIT and RT, are called the internal (state) variables (Maugin and Muschik, 1994; Coleman, 1967; Coleman and Gurtin, 1967; Muschik, 1990; Kestin and Rice, 1970). The related theory is called the internal variable theory (IVT). Physically, internal variables represent some microscopic material structural characteristics or their ensemble characteristics. For example, for materials of structural damages like microcracks, damage content tensor can be introduced as internal variables to supplement the description of a local equilibrium state besides the deformation and temperature. Internal variables are independent variables and obey a special rate equation (Maugin and Muschik, 1994; Muschik, 1990), where the rate of change of an internal variable is governed by the conjugated force that is defined as the partial derivative of the Helmholtz’s free energy with respect to the internal variable. These internal variables are “hidden” in the sense that their rate equations are local (i.e., spatially independent) and can be directly integrated (Maugin and Muschik, 1994; Muschik, 1990). In IVT, a concept of a local constrained nonequilibrium state is introduced. In other words, there always exists a local accompanying equilibrium (or quasi-equilibrium) state, onto which the local nonequilibrium state can be projected (Muschik, 1990). Although this mapping or projection may not be unique and one-to-one, the concepts of thermostatics are assumed applicable to the local accompanying equilibrium state, then to the corresponding local constrained nonequilibrium state. The dissipation (or internal power) induced by an internal variable is equal to the product of its rate of change and the corresponding conjugated force. The introduction of internal variables can be beneficial when selected material structural characteristics and their associated dissipations affect the local properties significantly (Capriz, 1989). However, other than the rate of change of an internal variable, irreversible processes associated with the evolutions of relevant material structural characteristics have to be considered. For example, irreversible processes such as the motions of microcracks and evolution of microcrack network, other than the rate of change of damage content tensor, are also crucial. To measure these irreversible processes, thermodynamic fluxes as in CIT are needed.

CIT and IVT have gone beyond equilibrium thermodynamics. However, they are limited to the validation of the assumption that there exists or accompanies a local equilibrium state. The applicability of the assumption of local equilibrium state relies on the validation of Boltzmann’s distribution (Prigogine, 1955; Fung, 1965). This restricts CIT and IVT only applicable to the cases which are only slightly fluctuated to the Boltzmann’s distribution and therefore not far away from thermodynamic equilibrium. The sufficient condition for the validity of the local equilibrium assumption is that all of the associated thermodynamic processes relax fast enough in comparison to some characteristic times of interest. For processes that have relaxation times, much larger than dynamic characteristic times of interest, they are away from thermodynamic equilibrium. However, they can be considered “locked in” a nearly equilibrium state at every instant. For processes that have relaxation times, much smaller than dynamic characteristic times of interest, they can be considered as instantaneous responses as in CIT. In these two situations, the assumption of a local equilibrium state is valid. However, in many cases, the characteristic relaxation times of the involved irreversible dissipation processes, either chemical, or mechanical, or thermal, or electric, can be at the same order of dynamic characteristic times of interest. Therefore, at any instant during these processes, every point in the continuum is at a state that is far away from thermodynamic equilibrium. The assumption of a local equilibrium state is no longer applicable.

Extended irreversible thermodynamics (EIT) relaxes the restriction of the assumption of a local equilibrium state by introducing a primitive concept of a local nonequilibrium state. Therefore, a local nonequilibrium state, which allows for far away from thermodynamic equilibrium, is assigned to every material point in a continuum at any instant (Jou et al., 1996, 1998; Müller and Ruggeri, 1998). EIT constructs a nonequilibrium state space in the following way. Besides the classical set of equilibrium state variables, an extended set of nonequilibrium state variables is introduced to complement the description of a local nonequilibrium state. These extended nonequilibrium state variables are also called thermodynamic fluxes, which are fast and nonconserved variables. Every irreversible dissipative process, either macroscopic (related to classical state variable) or microscopic (related to internal state variable), possesses an associated thermodynamic flux, which can be used to describe some types of interaction of a material point with its neighborhood. Each flux is activated by

the spatial gradients of either a classical state variable or an internal state variable, therefore, is spatially correlated. In contrast to the internal state variables, these thermodynamic fluxes are independent nonlocal state variables. In contrast to thermodynamic fluxes in CIT, these thermodynamic fluxes have relaxation characteristics and therefore directly contribute to state functions. These nonequilibrium contributions can be significant when relaxation times of thermodynamic fluxes are not negligible. For a higher-order approximation, the extended set of nonequilibrium state variables needs the introduction of high-order thermodynamic fluxes, which are the fluxes of the evolution processes of one-order-lower thermodynamic fluxes, and are therefore called the “fluxes of the fluxes”. In EIT, nonequilibrium state functions such as entropy, free energy and internal energy are functions of the entire set of state variables including both equilibrium and nonequilibrium variables. Furthermore, entropy flux depends on all thermodynamic fluxes, not just on heat flux as in the framework of CIT, RT or IVT. From the second law of EIT, the rate of entropy production is strictly non-negative to ensure the evolution direction of irreversible processes. This also imposes restrictions on the constitutive relations and evolution equations of thermodynamic fluxes.

### 1.3. Thermodynamics framework for this work

From the above discussions, for a general description of material damping, the introduction of a hybrid framework of internal variable theory and extended irreversible thermodynamics is necessary. In which, internal state variables are used to represent the microstructural characteristics, while extended state variables are used to describe the irreversible (macroscopic or microscopic) processes. However, for a typical simple material that we can neglect its microstructural characteristics, a framework of extended irreversible thermodynamics is sufficient. In this paper, we develop a damping model for simple materials in the framework of extended irreversible thermodynamics.

In the framework of EIT, introduction of thermodynamic fluxes and their associated relaxations lead to hyperbolic theories, which originally were motivated to resolve the unphysical ‘acausality’ feature of parabolic theories. As is discussed, hyperbolic theories become necessary whenever relaxation times of the irreversible processes are comparable to characteristic times of interest (Herrera and Pavón, 2002). This holds for gases, fluids and solids. For viscous processes, the range of relaxation times is quite wide: from  $10^{-13}$  s for ordinary fluids to  $10^2$  s for viscoelastic materials (Lebon et al., 1988). For gases and fluid, mechanism of viscosity is straightforward, but for solids it is complicated and of large variety. For structural metals, at relative low temperatures, internal friction associated with motion of dislocations (Granato and Lücke, 1956; Wilks, 1965) applies to damping. At elevated temperatures ( $T \sim 0.5T_{\text{melting}}$ ), viscous responses are substantial and attribute to various physical mechanisms (Norwick and Berry, 1972), such as viscous slip at grain boundaries (Zener, 1941; Kê, 1947) and rearrangement of pairs of atoms in an alloy (Zener, 1947). At higher temperatures, thermally activated dislocation mechanisms (Kê, 1949; Lakes and Quackebush, 1996) become the major contribution to dissipations. Therefore relaxation times of viscous processes depend not only on underlying viscous mechanism but also temperature. For example, for cold-worked copper, viscous relaxation times for dislocation motion vary from about 0.01 s at 65 K to 0.1  $\mu$ s at 135 K (Fig. 11 in Alers and Thompson, 1961); while for well-anneal copper, relaxation time is about 0.067  $\mu$ s at room temperature 300 K (Fig. 6 in Alers and Thompson, 1961). For polycrystalline aluminum, relaxation times of viscous slip of grain boundaries vary from about 2.5  $\mu$ s at 600 °C to 0.18 s at 285 °C (estimated from Eq. (23) in Kê, 1947). More recent works on internal friction of aluminum at elevated temperatures can be found in references by Jiang et al. (2005) and Cao and Huang (2004). For complex materials with strongly interacting constituents like some alloys or composites, relaxation in hierarchical stages suggests a (scale-invariant) distribution of relaxation times (Lakes and Quackebush, 1996; Norwick and Berry, 1961; Speake et al., 1999). Therefore, hyperbolic theories of relaxation times are necessary for treating viscous processes in solid materials for their general applications. We can see later on that hyperbolic theories naturally reduces to parabolic theories at relaxation times approach zeros.

Another aspect that attributes to damping is the irreversible thermal process. Thermal propagation processes consist of electron as well as phonon thermal conduction. The corresponding characteristic relaxations are determined by the times required for establishing thermal equilibrium between electrons and phonons. Typical characteristic relaxation time for phonon heat flux or electron heat flux is on the order of some tens of picoseconds or some tens of femtoseconds (Wall and Olsson, 1997; Pérez-Guerrero Noyola, 1997; Kostykin et al.,

1998). In general applications, thermal propagation process can be approximated as an instantaneous response; however for micro- and nanomechanical systems (MEMS and NEMS) (Lifshitz and Roukes, 2000; Herrera and Pavón, 2002) or for damping metallic structural material like magnesium alloys (Watanabe et al., 2004), thermoelastic relaxation times become not negligible due to the reduction of dimension in the former and reduction of thermal diffusivity in the latter. In this work, we used polycrystalline aluminum as the example structural material for the purpose of illustration of the model.

#### 1.4. Damping models

The most commonly applied material damping model is the linear viscous model, which lies in the category of CIT. For irreversible viscous process, the generalized flux is viscous stress and the associated generalized force is the rate of deformation. The coupling of thermal and mechanical fields is not considered in linear viscous model. In the category of CIT, another type of damping, i.e., the “thermoelastic damping”, is used by some authors to describe the Thomson effect (Zener, 1938). That is, when a solid is stressed adiabatically, there is an accompanying change in the temperature. In other words, mechanical and thermal fields are coupled. As a consequence of the second law of thermodynamics, irreversible heat transport occurs and entropy is produced when mechanical energy is converted into heat. Zener (1938) is the first person to study this type of damping mechanism. Boley (Boley and Weiner, 1960) has pointed out that for high frequency vibrations, thermoelastic damping can be significant. Many authors have extended Zener’s analysis to more general situations (Bishop and Kinra, 1993; Bishop et al., 1993; Adams and Bert, 1999). Thermoelastic damping model belongs to CIT and can be thought as the dynamic counterpart of the Duhem model (Parkus, 1968).

RT damping models, also called viscoelasticity models, usually involve integro-differential equations (Fabrizio and Morro, 1992; Drozdov and Kolmanovskii, 1994; etc.). The coupling and conversion between thermal and mechanical fields are usually not addressed in RT models. For materials having short memory, a simplified RT model can be obtained. In this special case, instead of the time history of the strain, additional state variables such as the rate and higher-order derivatives of deformation are introduced to determine the local state (Day, 1972; Truesdell, 1984). These additional state variables are not internal variables. They are not related to any internal material structures, and are both observable and controllable. In the category of IVT, by introducing one or more internal variables, Leisture et al. (Lesieutre and Mingori, 1990; Enlund and Lesieutre, 1999) have developed some viscoelastic models of first-order linear differential type. A finite element method (FEM) is employed for some applications.

In the framework of EIT, some models have been developed for rheological materials. Without considering thermal field, Lebon et al. (Lebon et al., 1988, 1991; Jou et al., 1996) developed a viscoelasticity (rheological) model. With considering thermal field, Pérez-Guerrero Noyola (1997) and Müller (Müller and Ruggeri, 1998) developed some rheological models for complex fluids. The development of the former models followed the EIT axioms, but used an CIT entropy flux instead of an EIT entropy flux. The latter started from generalized Gibbs’ equation and obtained evolution equations for thermodynamic fluxes, by assuming that entropy depends on a summation of quadratic inner products of heat flux, pressure and stress deviator.

Although “anelasticity” is inherently a thermodynamic phenomenon, various models that are not based on thermodynamic frameworks are proposed. It includes structural damping, hysteretic damping and modal damping (Nashif et al., 1985; Dominguea-Cascante and Jou, 1995; etc.). Most of these models are frequency-domain models. Some models can be applied in both time and frequency domains. Schiessel et al. (1995), Maia et al. (1998) and others have developed models by using fractional time derivatives in material constitutive equations. Chen and You (Chen and You, 1999) applied hereditary damping integral model, which led to an integro-differential governing equation.

#### 1.5. Contribution of this paper

In this paper, we have proposed a hybrid framework of IVT and EIT to develop general damping models for complex materials. For a typical simple material, a damping model is formulated in the framework of EIT. For such a typical system, heat flux and nonequilibrium stress are introduced as thermodynamic fluxes with respect to heat transport and viscous processes. From previously proposed EIT rheological models for com-

plex fluids (Lebon et al., 1988, 1991; Jou et al., 1996; Pérez-Guerrero Noyola, 1997; Müller and Ruggeri, 1998), these thermodynamic fluxes are adapted for a solid. In our formulation, instead of using the generalized Gibb's relation (Pérez-Guerrero Noyola, 1997; Müller and Ruggeri, 1998) or relaxing an EIT entropy flux to that of CIT (Lebon et al., 1991; Jou et al., 1996), we start from assuming some first-order relaxation equations for the evolutions of thermodynamic fluxes. This method gives us more freedom to describe the interactions between the nonequilibrium thermal and mechanical fields. In our model, coupling is described in both equilibrium and nonequilibrium mechanical and thermal fields. This means that evolution equation for a thermodynamic flux is related not only to its own generalized force but also to other generalized forces. In this, some mechanisms of damping can be clearly described. Based on the concept of EIT state function and using the evolution equation of EIT entropy, we explicitly derived the expressions for specific internal energy, entropy and free energy as a function of local nonequilibrium state. Then a nonlinear energy balance equation, which governs the interchanges of thermal and mechanical energies, is derived to complete the formulation. Various limiting cases of EIT damping model are investigated and related to typical CIT models like thermoelastic, linear viscous and visco-thermoelastic models.

We used the developed model to study longitudinal vibrations of a rod of uniform cross section. Due to nonlinearity involved in the developed model, various techniques are employed for the solutions. The relations of damping ratio (or dissipation characteristics) to natural (or driving) frequency are investigated. The effects of nonequilibrium parameters on dissipation characteristics, like characteristic relaxation times of thermodynamic fluxes and the interactions between nonequilibrium fields, are investigated. In comparison, a CIT visco-thermoelastic model and an IVT damping model are presented (or developed), and numerical results are given.

## 2. Nonequilibrium thermodynamics damping model

During the vibrations of a dissipative system, various coupled irreversible processes can be involved. Let us consider a material point in the system. Under the assumption of local equilibrium state, interactions between the material point and its neighborhood is described by instantaneous responses. However, this restriction has to be removed at situations when these interactions cannot relax completely during the characteristic time-periods of interest. We can accommodate (or classify) these interactions by (or into) various irreversible processes of relaxation characteristics. For each irreversible process, we introduced a thermodynamic flux as the extended state variables. Therefore, to describe a local nonequilibrium state of a material point, we have a set of classical state variables and a set of extended state variables. According to informational statistical mechanics, thermodynamic fluxes can be assumed in an evolution of first-order relaxation (Garcis-Colin et al., 1994). The “causes” for their evolutions are the gradient of the associated state variables (Onsager, 1931a,b). For a general dissipative system, the inclusion of irreversible processes at macroscopic levels is sufficient. However, physically, all of the dissipation mechanisms fundamentally depend on the evolution processes of some material microstructures. For some dynamic systems, dissipation mechanisms rely significantly on some specific characteristics of material architecture. In these cases, internal state variables are needed to describe these material microstructural characteristics. Any irreversible processes, related to the evolution processes of these microstructural characteristics, need the introduction of the corresponding thermodynamic fluxes. For example, domain-wall motions contribute to the dissipation of a ferromagnetic or piezoelectric material. The characteristics of domains, in a more accurate model, then, have to be described by internal state variables (Lu and Hanagud, 2006). The associated domain-wall evolution has to be described by some thermodynamic fluxes.

In this paper, we only restrict our discussions to a typical dynamic system, where the introduction of internal variables is assumed not needed. Only mechanical and thermal fields are involved. The irreversible processes of consideration consist of only viscous and heat transport processes. We introduce nonequilibrium stresses  $\sigma_{ij}^{nc}$  and heat flux  $q_i$  as the corresponding thermodynamic fluxes. Therefore, the set of classical state variables is:  $\{u_i(\text{or } \varepsilon_{ij}), v_i(\text{or } \dot{\varepsilon}_{ij}), \theta(\text{or } e)\}$ , where  $u_i$  is displacement,  $\varepsilon_{ij}$  the strain,  $v_i$  the velocity,  $\dot{\varepsilon}_{ij}$  the strain rate,  $\theta$  the absolute temperature, and  $e$  the specific internal energy. The set of extended state variables is:  $\{\sigma_{ij}^{nc}, q_i\}$ . We further assume that the total stress  $\sigma_{ij}$  is a summation of an elastic stresses  $\sigma_{ij}^e$  and nonequilibrium (or viscous) stresses  $\sigma_{ij}^{nc}$ . In the following, we first formulate the conservation equations and constitutive equations. Then based on the concept of state functions in the framework of EIT, we derive the expressions for state functions, such as specific internal energy  $e$ , specific entropy  $s$  and specific Helmholtz's free energy  $f$ , by applying the 1st

and 2nd laws of thermodynamics. The constitutive equations consists of two parts: equilibrium constitutive relations and nonequilibrium constitutive relations. In this paper, the discussion is limited to infinitesimal small deformations. However, generalization to large deformations should not raise fundamental difficulties.

(a) Conservation of linear momentums:

$$\rho \dot{v}_i = \sigma_{ij,j} + \rho X_i \quad (1)$$

(b) Conservation of internal energy:

$$\rho \dot{e} = -q_{i,i} + \sigma_{ij} \dot{\epsilon}_{ij} \quad (2)$$

Equilibrium constitutive relations:

(a) Thermoelastic constitutive relations (or the Duhamel–Neumann law):

$$\sigma_{ij}^e = C_{ijkl} \epsilon_{kl} - \beta_{ij} (\theta - \theta_0) \quad (3)$$

Nonequilibrium constitutive relations – evolutions of thermodynamic fluxes:

(a) Evolution of heat flux:

$$\tau_q \dot{q}_i + q_i = -k_{ij} \theta_{,j} + A_{ijk} \dot{\epsilon}_{jk} + B_{ijk} q_{j,k} + k_{ijkl}^{\sigma} \sigma_{jk,l}^{\text{ne}} \quad (4)$$

(b) Evolution of nonequilibrium (or viscous) stresses:

$$\tau_{\sigma} \dot{\sigma}_{ij}^{\text{ne}} + \sigma_{ij}^{\text{ne}} = A_{ijk} \theta_{,k} + \eta_{ijkl} \dot{\epsilon}_{kl} + k_{ijkl}^{\sigma} q_{k,l} + F_{ijklm} \sigma_{kl,m}^{\text{ne}} \quad (5)$$

Kinematic relations:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (6)$$

Total stresses:

$$\sigma_{ij} = \sigma_{ij}^e + \sigma_{ij}^{\text{ne}} \quad (7)$$

In these equations,  $\rho$  is the mass density,  $C_{ijkl}$  the elastic moduli,  $\beta_{ij}$  the thermal expansion moduli,  $X_i$  the external body force,  $k_{ij}$  the heat conduction coefficients,  $\theta_0$  the initial temperature,  $\eta_{ijkl}$  the viscous moduli,  $B_{ijkl}$  the heat transport coefficients, and  $F_{ijklm}$  the viscosity transport coefficients.  $A_{ijk}$  and  $k_{ijkl}^{\sigma}$  are the thermal-viscous coefficients.  $\tau_q$  and  $\tau_{\sigma}$  are the relaxation times for heat flux and nonequilibrium stresses, respectively. As shown in Eqs. (4) and (5), thermoelastic coefficients  $\beta_{ij}$  describe the coupling between equilibrium thermal and mechanical interaction; while the thermal-viscous coefficients  $k_{ijkl}^{\sigma}$  describe the coupling between nonequilibrium mechanical and thermal fields. According to the Curie's principle of symmetry (Fung, 1965),  $A_{ijk}$  should vanish. For an isotropic material,

$$\begin{aligned} C_{ijkl} &= \lambda \delta_{kl} \delta_{ij} + 2\mu \delta_{ik} \delta_{jl}; & \beta_{ij} &= \beta \delta_{ij}; & k_{ij} &= k \delta_{ij}; & B_{ijk} &= 0; & F_{ijklm} &= 0 \\ k_{ijkl}^{\sigma} &= k_1^{\sigma} \delta_{kl} \delta_{ij} + k_2^{\sigma} \delta_{ik} \delta_{jl} + k_3^{\sigma} \delta_{il} \delta_{jk}; & \eta_{ijkl} &= 2\eta \delta_{ik} \delta_{jl} \end{aligned} \quad (8)$$

in which  $\delta_{pq}$  is the Kronecker Delta,  $\lambda$  and  $\mu$  the Lamé's constants,  $k$  a heat conduction constant,  $\beta$  a thermal expansion constant,  $\eta$  a viscous constant, and  $k_i^{\sigma}$  ( $i = 1, 2, 3$ ) are the thermal-viscous constants.

### 2.1. Expression for the rate of entropy production $\sigma^s$

In the framework of irreversible thermodynamics, specific entropy  $s$  is assumed to govern the local synthesized process and obey a first-order time evolution law. The evolution equation of entropy is given by

$$\rho \dot{s} = -J_{i,i}^s + \sigma^s \quad (9)$$

where  $s$  is the specific entropy,  $J_i^s$  the entropy flux, and  $\sigma^s$  the rate of entropy production. Entropy flux is defined as the entropy flow to and/or from its surrounding. In contrast to CIT, RT and IVT, where entropy flux only depends on the heat flux; in EIT, entropy flux depends on all of the thermodynamic fluxes.

$$J_i^s = J_i^s(\theta, q_j, \sigma_{kl}^{nc}) \tag{10}$$

The specific Helmholtz's free energy density  $f$  is defined by  $f = e - \theta s$ . Specific entropy, specific internal energy and specific free energy are a function of local nonequilibrium state.

$$\zeta = \zeta(\theta, \varepsilon_{ij}, q_k, \sigma_{lm}^{nc}), \text{ where } \zeta \text{ denotes symbol } e, s, \text{ or } f \tag{11}$$

In thermodynamics, the 2nd law states that the rate of entropy production  $\sigma^s$  is a nonnegative quantity.

$$\sigma^s \equiv \rho \dot{s} + J_{i,i}^s \geq 0 \tag{12}$$

From the definition of specific Helmholtz's free energy, we obtain

$$\theta \dot{s} = \dot{e} - \dot{f} - s \dot{\theta} \tag{13}$$

By multiplying Eq. (11) by  $\theta$  and substituting Eq. (13) into it, we obtain

$$\theta \sigma^s \equiv \rho \dot{e} - \rho \dot{f} - \rho s \dot{\theta} + \theta J_{i,i}^s \geq 0 \tag{14}$$

From Eq. (11), taking the time derivative of specific free energy with respect to time, we can obtain

$$\dot{f} = \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial \sigma_{ij}^{nc}} \dot{\sigma}_{ij}^{nc} \tag{15}$$

From Eq. (10), the divergence of the entropy flux, up to the 1st order approximation, is obtained.

$$J_{i,i}^s = A_{ij}^1 q_{j,i} + A_i^2 \theta_{,i} + A_{ijk}^3 \sigma_{jk,i}^{nc} \tag{16}$$

where  $A_{ij}^1 = \frac{\partial J_i^s}{\partial q_j}$ ;  $A_i^2 = \frac{\partial J_i^s}{\partial \theta}$ ;  $A_{ijk}^3 = \frac{\partial J_i^s}{\partial \sigma_{jk}^{nc}}$

For an isotropic material, substituting Eqs. (15), (16), (3), (4) and (5) into (14) and collecting terms, yields,

$$\begin{aligned} \theta \sigma^s = & -\rho \left( \frac{\partial f}{\partial \theta} + s \right) \dot{\theta} - \left( \rho \frac{\partial f}{\partial \varepsilon_{ij}} + \rho \frac{2\eta}{\tau_\sigma} \frac{\partial f}{\partial \sigma_{ij}^{nc}} - \sigma_{ij} \right) \dot{\varepsilon}_{ij} + \left( \rho \frac{k}{\tau_q} \frac{\partial f}{\partial q_i} + \theta A_i^2 \right) \dot{\theta}_{,i} \\ & + \left[ -\delta_{ij} + \theta A_{ji}^1 - \frac{\rho}{\tau_\sigma} \left( k_1^\sigma \frac{\partial f}{\partial \sigma_{kk}^{nc}} \delta_{ij} + k_2^\sigma \frac{\partial f}{\partial \sigma_{ij}^{nc}} + k_3^\sigma \frac{\partial f}{\partial \sigma_{ji}^{nc}} \right) \right] q_{i,j} \\ & + \left[ -\frac{\rho}{\tau_q} \left( k_1^\sigma \frac{\partial f}{\partial q_i} \delta_{jk} + k_2^\sigma \frac{\partial f}{\partial q_j} \delta_{ik} + k_3^\sigma \frac{\partial f}{\partial q_k} \delta_{ij} \right) + \theta A_{kij}^3 \right] \sigma_{ij,k}^{nc} + \frac{\rho}{\tau_q} \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\rho}{\tau_\sigma} \frac{\partial f}{\partial \sigma_{ij}^{nc}} \dot{\sigma}_{ij}^{nc} \geq 0 \end{aligned} \tag{17}$$

Assume that the rate of entropy production  $\sigma^s$  is a function of  $\theta$ ,  $q_j$ , and  $\sigma_{ij}^{nc}$ . Sufficient conditions to impose the inequality in (17) for any situations are as follows:



$$\frac{\partial f}{\partial \theta} + s = 0 \tag{18a}$$

$$\rho \frac{\partial f}{\partial \varepsilon_{ij}} + \rho \frac{2\eta}{\tau_\sigma} \frac{\partial f}{\partial \sigma_{ij}^{nc}} - \sigma_{ij} = 0 \tag{18b}$$

$$\rho \frac{k}{\tau_q} \frac{\partial f}{\partial q_i} + \theta \Lambda_i^2 = 0 \tag{18c}$$

$$-\delta_{ij} + \theta \Lambda_{ji}^1 - \frac{\rho}{\tau_\sigma} \left( k_1^\sigma \frac{\partial f}{\partial \sigma_{kk}^{nc}} \delta_{ij} + k_2^\sigma \frac{\partial f}{\partial \sigma_{ij}^{nc}} + k_3^\sigma \frac{\partial f}{\partial \sigma_{ji}^{nc}} \right) = 0 \tag{18d}$$

$$-\frac{\rho}{\tau_q} \left( k_1^\sigma \frac{\partial f}{\partial q_i} \delta_{jk} + k_2^\sigma \frac{\partial f}{\partial q_j} \delta_{ik} + k_3^\sigma \frac{\partial f}{\partial q_k} \delta_{ij} \right) + \theta \Lambda_{kij}^3 = 0 \tag{18e}$$

$$\frac{\partial f}{\partial q_i} = \hat{\alpha}(\theta) q_i \tag{18f}$$

$$\frac{\partial f}{\partial \sigma_{ij}^{nc}} = \hat{\beta}(\theta) \sigma_{ij}^{nc}; \quad \hat{\alpha} \geq 0; \quad \hat{\beta} \geq 0 \tag{18g}$$

Substituting the relations (18) into (17), we obtain

$$\theta \sigma^s = \frac{\rho}{\tau_q} \hat{\alpha}(\theta) q_i q_i + \frac{\rho}{\tau_\sigma} \hat{\beta}(\theta) \sigma_{ij}^{nc} \sigma_{ij}^{nc} \geq 0 \tag{19}$$

From Eq. (18a), we have

$$\frac{\partial f}{\partial \theta} = -s \tag{20}$$

By considering Eqs. (7) and (18b), we can assume

$$\frac{\partial f}{\partial \varepsilon_{ij}} = \frac{1}{\rho} \sigma_{ij}^e; \quad \frac{\partial f}{\partial \sigma_{ij}^{nc}} = \frac{1}{\rho} \frac{\tau_\sigma}{2\eta} \sigma_{ij}^{nc} \tag{21}$$

From Eqs. (18g) and (21b), we have

$$\hat{\beta}(\theta) = \frac{1}{\rho} \frac{\tau_\sigma}{2\eta} \tag{22}$$

Substituting Eq. (21) into (18d), yields,

$$\Lambda_{ij}^1 = \frac{1}{\theta} \delta_{ij} + \frac{1}{2\eta\theta} (k_1^\sigma \sigma_{kk}^{nc} \delta_{ij} + k_2^\sigma \sigma_{ji}^{nc} + k_3^\sigma \sigma_{ij}^{nc}) \tag{23}$$

Substituting Eq. (18f) into (18c) and (18e), gives,

$$\Lambda_i^2 = -\rho \frac{\hat{\alpha}}{\theta} \frac{k}{\tau_q} q_i \tag{24a}$$

$$\Lambda_{ijk}^3 = \rho \frac{\hat{\alpha}}{\theta} \frac{1}{\tau_q} (k_1^\sigma q_j \delta_{ik} + k_2^\sigma q_k \delta_{ij} + k_3^\sigma q_i \delta_{jk}) \tag{24b}$$

From Eq. (10), we can write the divergence of entropy flux as follows

$$\begin{aligned} J_{i,i}^s &= \left( \frac{1}{\theta} q_{i,i} - \rho \frac{\hat{\alpha}}{\theta} \frac{k}{\tau_q} q_i \theta_{,i} \right) + \frac{1}{2\eta\theta} (k_1^\sigma \sigma_{kk}^{nc} \delta_{ij} + k_2^\sigma \sigma_{ji}^{nc} + k_3^\sigma \sigma_{ij}^{nc}) q_{j,i} \\ &\quad + \rho \frac{\hat{\alpha}}{\theta} \frac{1}{\tau_q} (k_1^\sigma q_j \delta_{ik} + k_2^\sigma q_k \delta_{ij} + k_3^\sigma q_i \delta_{jk}) \sigma_{jk,i}^{nc} \end{aligned} \tag{25}$$

As nonequilibrium stress  $\sigma_{ij}^{nc}$  reduces to zero, the divergence of entropy flux should reduce to that of CIT, which is  $(\frac{q_i}{\theta})_{,i}$ . Observing that the first term in Eq. (25), it will match the  $(\frac{q_i}{\theta})_{,i}$ , if and only if

$$\hat{\alpha} = \frac{\tau_q}{\rho k \theta} \tag{26}$$

Therefore, from Eqs. (19), (22) and (26), we can obtain the divergence of entropy flux and the rate of entropy production.

$$\begin{cases} J_{i,i}^s = \left(\frac{q_i}{\theta}\right)_{,i} + \frac{1}{2\eta\theta} (k_1^\sigma \sigma_{kk}^{ne} \delta_{ij} + k_2^\sigma \sigma_{ji}^{ne} + k_3^\sigma \sigma_{ij}^{ne}) q_{j,i} + \frac{1}{k\theta^2} (k_1^\sigma q_j \delta_{ik} + k_2^\sigma q_k \delta_{ij} + k_3^\sigma q_i \delta_{jk}) \sigma_{jk,i}^{ne} \\ \sigma^s = \frac{1}{2\eta\theta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{1}{k\theta^2} q_i q_i \end{cases} \tag{27}$$

The 2nd law of thermodynamics as stated in Eq. (19) imposes that  $\eta \geq 0$  and  $k \geq 0$ .

### 2.2. Specific internal energy

The total derivative of specific free energy can be obtained by substituting Eqs. (20), (21), (18f) and (26) into (15),

$$df = -s d\theta + \frac{1}{\rho} \sigma_{ij}^e d\varepsilon_{ij} + \frac{1}{\rho} \frac{\tau_\sigma}{2\eta} \sigma_{ij}^{ne} d\sigma_{ij}^{ne} + \frac{1}{\rho} \frac{\tau_q}{k} \left(\frac{q_i}{\theta}\right) dq_i \tag{28}$$

We substitute Eqs. (3) and (8) into (28), and collect terms.

$$df = \left(-s + \frac{1}{\rho} \beta \varepsilon + \frac{1}{\rho} \frac{\tau_q}{2k\theta^2} q_i q_i\right) d\theta + \frac{1}{\rho} d\left(\frac{\lambda}{2} \varepsilon^2 + \mu \varepsilon_{ij} \varepsilon_{ij} - \beta(\theta - \theta_0) \varepsilon + \frac{\tau_\sigma}{4\eta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{\tau_q}{2k\theta} q_i q_i\right) \tag{29}$$

Since the Helmholtz's free energy is a function of state, Eq. (29) can be integrated into

$$f = \frac{1}{\rho} \left(\frac{\lambda}{2} \varepsilon^2 + \mu \varepsilon_{ij} \varepsilon_{ij} - \beta(\theta - \theta_0) \varepsilon + \frac{\tau_\sigma}{4\eta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{\tau_q}{2k\theta} q_i q_i\right) + f_1(\theta, \varepsilon_{ij}, q_i, \sigma_{ij}^{ne}) \tag{30}$$

where  $f_1$  is a function to be determined. From Eq. (30), we calculate the partial derivatives of  $f$  with respect to state variables  $q_i$ ,  $\sigma_{ij}^{ne}$ , and  $\varepsilon_{ij}$ .

$$\frac{\partial f}{\partial q_i} = \frac{1}{\rho} \frac{\tau_q}{k\theta} q_i + \frac{\partial f_1}{\partial q_i} \tag{31a}$$

$$\frac{\partial f}{\partial \sigma_{ij}^{ne}} = \frac{1}{\rho} \frac{\tau_\sigma}{2\eta} \sigma_{ij}^{ne} + \frac{\partial f_1}{\partial \sigma_{ij}^{ne}} \tag{31b}$$

$$\frac{\partial f}{\partial \varepsilon_{ij}} = \frac{1}{\rho} (\lambda \varepsilon \delta_{ij} + 2\mu \varepsilon_{ij} - \beta(\theta - \theta_0) \delta_{ij}) + \frac{\partial f_1}{\partial \varepsilon_{ij}} = \frac{1}{\rho} \sigma_{ij}^e + \frac{\partial f_1}{\partial \varepsilon_{ij}} \tag{31c}$$

Comparing Eqs. (31a)–(31c) with (21) and (18f) and considering (26), we can conclude

$$\frac{\partial f_1}{\partial q_i} = \frac{\partial f_1}{\partial \sigma_{ij}^{ne}} = \frac{\partial f_1}{\partial \varepsilon_{ij}} = 0 \tag{32}$$

Therefore, the  $f_1$  reduces to a function of only temperature  $\theta$ . Then we can denote  $f_1(\theta, \varepsilon_{ij}, q_i, \sigma_{ij}^{ne})$  as a  $C(\theta)$ .

$$\rho f = \frac{\lambda}{2} \varepsilon^2 + \mu \varepsilon_{ij} \varepsilon_{ij} - \beta(\theta - \theta_0) \varepsilon + \frac{\tau_\sigma}{4\eta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{\tau_q}{2k\theta} q_i q_i + C(\theta) \tag{33}$$

It is noted that as relaxation times  $\tau_\sigma$  and  $\tau_q$  reduce to zero, specific free energy  $f$  in Eq. (33) reduces to that of CIT (Fung, 1965). Substituting Eq. (33) into (20), we can obtain the entropy.

$$\rho s = -\rho \frac{\partial f}{\partial \theta} = -\frac{dC}{d\theta} + \beta \varepsilon + \frac{\tau_q}{2k} \frac{q_i}{\theta} \frac{q_i}{\theta} \tag{34}$$

Then, specific internal energy can be obtained by using the definition of specific Helmholtz's free energy  $f$ .

$$\rho e = \rho(f + s\theta) = \frac{\lambda}{2} \varepsilon^2 + \mu \varepsilon_{ij} \varepsilon_{ij} + \beta \theta_0 \varepsilon + \frac{\tau_\sigma}{4\eta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{\tau_q}{k\theta} q_i q_i + \left[ C(\theta) - \theta \frac{dC}{d\theta} \right] \tag{35}$$

Similarly, from Eq. (35), we can see that the form of CIT internal energy  $e$  can be recovered in the limit case of EIT, i.e., as relaxation times  $\tau_\sigma$  and  $\tau_q$  reduce to zero. Considering zero strain, heat flux and viscous stress, from Eqs. (3) and (34), we can obtain the following relation.

$$\rho de = dQ = -\theta \frac{d^2 C}{d\theta^2} d\theta \tag{36}$$

On the other hand, from the definition of the specific heat, at zero strain

$$dQ = \rho c_v(\theta) d\theta \tag{37}$$

where  $c_v(\theta)$  is the specific heat. Hence

$$c_v(\theta) = -\frac{\theta}{\rho} \frac{d^2 C}{d\theta^2} \tag{38}$$

Then, we have

$$\left[ C(\theta) - \theta \frac{dC}{d\theta} \right]_\theta = \left[ C(\theta) - \theta \frac{dC}{d\theta} \right]_{\theta_0} + \int_{\theta_0}^\theta \left( -\frac{\theta}{\rho} \frac{d^2 C}{d\theta^2} \right) d\theta = e_0 + \int_{\theta_0}^\theta \rho c_v(\theta) d\theta \tag{39}$$

where  $e_0$  is the reference value of internal energy  $e$  at  $\theta_0$ . Then, we can obtain the energy balance equation that governs the interchange of energies by substituting Eqs. (35) and (39) into (3).

$$\left( \rho c_v(\theta) - \frac{\tau_q}{k\theta^2} q_i q_i \right) \dot{\theta} = -q_{i,i} + \sigma_{ij}^{ne} \dot{\epsilon}_{ij} - \beta \theta \dot{\epsilon} - \frac{\tau_\sigma}{2\eta} \sigma_{ij}^{ne} \dot{\sigma}_{ij}^{ne} - \frac{2\tau_q}{k\theta} q_i \dot{q}_i \tag{40}$$

### 2.3. Summary of governing equations

For an isotropic material, the governing equations can be obtained from Eqs. (2), (4), (5) and (40).

$$\begin{cases} \rho \ddot{u}_i = \lambda \epsilon_{,i} + 2\mu \epsilon_{ij,j} - \beta T_{,i} + \sigma_{ij,j}^{ne} + \rho X_i \\ \tau_\sigma \dot{\sigma}_{ij}^{ne} = k_1^\sigma q_{k,k} \delta_{ij} + k_2^\sigma q_{i,j} + k_3^\sigma q_{j,i} + 2\eta \dot{\epsilon}_{ij} - \sigma_{ij}^{ne} \\ \tau_q \dot{q}_i = -kT_{,i} + k_1^\sigma \sigma_{ij,j}^{ne} + k_2^\sigma \sigma_{ji,i}^{ne} + k_3^\sigma \sigma_{jj,i}^{ne} - q_i \\ \left( \rho c_v - \frac{\tau_q}{k\theta^2} q_i q_i \right) \dot{T} = -q_{i,i} + \sigma_{ij}^{ne} \dot{\epsilon}_{ij} - \beta \theta \dot{\epsilon} - \frac{\tau_\sigma}{2\eta} \sigma_{ij}^{ne} \dot{\sigma}_{ij}^{ne} - \frac{2\tau_q}{k\theta} q_i \dot{q}_i, \quad \text{where } T = \theta - \theta_0 \end{cases} \tag{41}$$

### 2.4. Propagations of thermal and viscous disturbances

We can derive the nonlinear wavelike propagation equations for  $\theta$ ,  $q_i$  and  $\sigma_{ij}^{ne}$ , up to the 2nd order nonlinearity. Taking a spatial derivative of Eq. (41d) and substituting it into the equation that is obtained by taking time derivative of (41c), we can obtain the propagation equation for  $q_i$  in (42a). Similarly, taking a spatial derivative of Eq. (41c) and substituting it into the equation that is obtained by taking time derivative of (41d), we can obtain the propagation equation for  $\theta$  in (42b). Taking the spatial derivative of (41a) and substituting it into the equation that is obtained by taking time derivative of (41b), we can obtain a propagation equation for  $\sigma_{ij}^{ne}$  in (42c).

$$\begin{cases} \ddot{q}_i + \frac{1}{\tau_q} \dot{q}_i - c^2 q_{i,jj} = c^2 e_{ijk} e_{jpk} q_{p,qk} - c^2 \left( \sigma_{kl}^{ne} \dot{\epsilon}_{kl} - \beta \theta \dot{\epsilon} - \frac{\tau_\sigma}{2\eta} \sigma_{kl}^{ne} \dot{\sigma}_{kl}^{ne} - \frac{2\tau_q}{k\theta_0} q_k \dot{q}_k \right)_{,i} \\ \quad + \frac{1}{\tau_q} (k_1^\sigma \dot{\sigma}_{ij,j}^{ne} + k_2^\sigma \dot{\sigma}_{ji,i}^{ne} + k_3^\sigma \dot{\sigma}_{jj,i}^{ne}) \\ \ddot{\theta} - c^2 \theta_{,ii} = \frac{1}{\rho c_v} (\sigma_{kl}^{ne} \dot{\epsilon}_{kl} - \beta \theta \dot{\epsilon} - \frac{\tau_\sigma}{2\eta} \sigma_{kl}^{ne} \dot{\sigma}_{kl}^{ne} - \frac{2\tau_q}{k\theta_0} q_k \dot{q}_k)_{,t} + \frac{1}{\rho c_v \tau_q} (q_{i,i} - k_1^\sigma \sigma_{ij,ji}^{ne} - k_2^\sigma \sigma_{ji,ji}^{ne} - k_3^\sigma \sigma_{jj,ii}^{ne}) \\ \ddot{\sigma}_{ij}^{ne} + \frac{1}{\tau_\sigma} \dot{\sigma}_{ij}^{ne} - d^2 (\sigma_{ik,kj}^{ne} + \sigma_{jk,ki}^{ne}) = \frac{k_1^\sigma}{\tau_\sigma} \dot{q}_{k,k} \delta_{ij} + \frac{k_2^\sigma}{\tau_\sigma} (\dot{q}_{i,j} + \dot{q}_{j,i}) + \frac{2\eta \dot{\epsilon}}{\rho \tau_\sigma} \epsilon_{,ij} \\ \quad + \frac{2\eta \mu}{\rho \tau_\sigma} (\epsilon_{ik,kj} + \epsilon_{jk,ki}) - \frac{2\eta \beta}{\rho \tau_\sigma} T_{,ij} + \frac{2\eta}{\tau_\sigma} (X_{i,j} + X_{j,i}) \end{cases} \tag{42}$$

where  $e_{ijk}$  is the permutation symbol used in tensor analysis,  $c^2 = k/(\rho c_v \tau_q)$ , and  $d^2 = 2\eta/(\rho \tau_\sigma)$ . We can see that thermal disturbance (temperature or heat flux) propagate at a speed  $c$ , while the viscous disturbance propagates at a speed  $d$ . Since relaxation time  $\tau_q$  is usually quite small, e.g., at the order of  $10^{-20}$ s (Pérez-Guerrero Noyola, 1997), it leads to a very large propagation speed  $c$ . However, relaxation time  $\tau_\sigma$  can vary in a wide range: from  $10^{-13}$  s in ordinary fluids to  $10^2$  s in viscoelastic materials (Lebon et al., 1988). Usually, the viscous propagation speed  $d$  is much smaller than thermal propagation speed  $c$ . If relaxation times  $\tau_q$  and  $\tau_\sigma$  tend to zero, evolution equations for heat flux and nonequilibrium stresses degenerate to classical Fourier’s Law and linear viscous equation, respectively. Then the wavelike propagation equations for thermal and viscous disturbances in (44) reduce to some parabolic equations of infinite propagation speeds.

2.5. Limiting cases of the EIT damping model

Limiting case 1: Viscous-thermoelastic model

If relaxation times  $\tau_q$  and  $\tau_\sigma$  are zeros, the EIT damping model reduces to a CIT model. Especially if thermal-viscous coefficients  $k_{ijkl}^\sigma$  are zeros, i.e., no thermal-viscous coupling, a CIT viscous-thermoelastic model is obtained.

$$\begin{cases} \rho \ddot{u}_i = \lambda \varepsilon_{,i} + 2\mu \varepsilon_{ij,j} - \beta T_{,i} + \sigma_{ij,j}^{ne} + \rho X_i \\ \sigma_{ij}^{ne} = 2\eta \dot{\varepsilon}_{ij} \\ q_i = -kT_{,i} \\ \rho c_v \dot{T} = -q_{i,i} + \sigma_{ij}^{ne} \dot{\varepsilon}_{ij} - \beta \theta \dot{\varepsilon} \end{cases} \quad (43)$$

Limiting case 2: Thermoelastic model

If viscous field is negligible, i.e., nonequilibrium stresses  $\sigma_{ij}^{ne}$  are zeros, the viscous-thermoelastic model in (43) reduces to a CIT thermoelastic model.

$$\begin{cases} \rho \ddot{u}_i = \lambda \varepsilon_{,i} + 2\mu \varepsilon_{ij,j} - \beta T_{,i} + \rho X_i \\ q_i = -kT_{,i} \\ \rho c_v \dot{T} = -q_{i,i} - \beta \theta \dot{\varepsilon} \end{cases} \quad (44)$$

Limiting case 3: Linear viscous model

If thermal field is negligible, i.e., temperature fluctuations and heat flux are zeros, the viscous-thermoelastic model in (43) reduces to a CIT linear viscous model.

$$\begin{cases} \rho \ddot{u}_i = \lambda \varepsilon_{,i} + 2\mu \varepsilon_{ij,j} + \sigma_{ij,j}^{ne} + \rho X_i \\ \sigma_{ij}^{ne} = 2\eta \dot{\varepsilon}_{ij} \end{cases} \quad (45)$$

For the former two cases, where a thermal field is considered, the conversion between mechanical and thermal energies is governed by a nonlinear energy balance equation (see Eqs. (43d) and (44c)).

2.6. Discussions on EIT damping model

If thermal field is negligible, the stress and strain relations in Eqs. (3) and (5) gives us a Poynting–Thomson model or a Zener (standard) model (Fig. 1a). If relaxation time  $\tau_\sigma$  is zero, the Poynting–Thomson model reduces to a Kelvin–Voigt model. If elastic stresses are negligible, a Maxwell model is obtained. To characterize a more complicated viscosity phenomenon, we can use multiple irreversible viscous processes. Each of them has a corresponding nonequilibrium (viscous) stresses, which bears its own relaxation time and viscous coefficients. This complex model is presented in Fig. 1b. However, in order to cope with other possible dissipation mechanisms, such as independent strain relaxation phenomenon and many other empirically derived viscous models observed from stress–strain relations, we need the introduction of the internal state variables

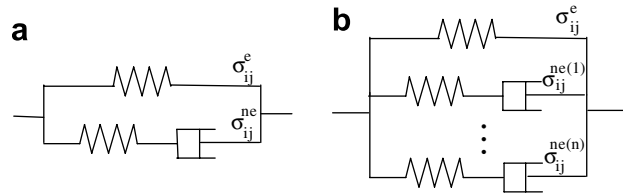


Fig. 1. Viscosity models (a) Poynting–Thomson model; (b) generalized complex model.

to accommodate the corresponding material microstructures and their evolutions, like inelastic strain. In other words, we need to use a hybrid thermodynamics framework of IVT and EIT.

2.7. Damping capacity

Specific internal energy can be decomposed into two parts: an elastic strain energy  $e^e$  and a nonequilibrium internal energy  $e^{ne}$ , i.e.,

$$\begin{cases} \rho e^e = \frac{1}{2} \sigma_{ij}^e (\epsilon_{ij} - \alpha T \delta_{ij}) = \frac{1}{2} \epsilon^2 + \mu \epsilon_{ij} \epsilon_{ij} - \beta T \epsilon + \frac{3}{2} \beta \alpha T^2 \\ \rho e^{ne} = -\frac{3}{2} \beta \alpha T^2 + \beta \theta \epsilon + \frac{\tau_a}{4\eta} \sigma_{ij}^{ne} \sigma_{ij}^{ne} + \frac{\tau_a}{k\theta} q_i q_i + C(\theta) - \theta \frac{dC}{d\theta} \end{cases} \quad (46)$$

where  $\beta = (3\lambda + 2\mu)\alpha$ ;  $\alpha$  the thermal expansion coefficient. For cyclic forced vibrations, the change of local nonequilibrium internal energy per cycle is defined by

$$\Delta W = \oint \rho \dot{e}^{ne} dt \quad (47)$$

If  $W$  is the local maximum strain energy during one cycle, the local specific damping capacity  $\Delta\psi$  is defined by

$$\Delta\psi = \frac{\Delta W}{W} \quad (48)$$

The volume averaged specific damping capacity  $\Psi$  is defined as the ratio of total energy dissipation per cycle to strain energy, i.e.,

$$\Psi = \frac{\int_V \Delta W dv}{\int_V W dv} \quad (49)$$

3. Applications: Longitudinal vibrations of a rod

3.1. Governing equations

Consider an isotropic rod undergoing longitudinal vibrations (Fig. 2). A 1-D stress problem can be addressed. Therefore, only one out of the six stress components, i.e.,  $\sigma_{xx}$ , is nontrivial. Assuming that shearing

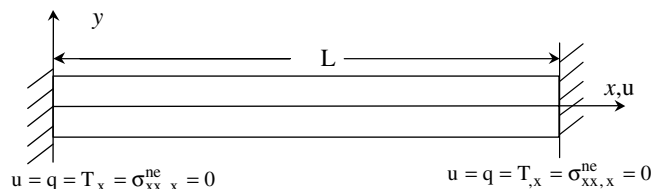


Fig. 2. A rod of uniform cross section.

effects are negligible, we have three normal strain components  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ , and  $\epsilon_{zz}$ . For heat flux, only  $x$ -component, denoted by ‘ $q$ ’, is nontrivial. These state variables only spatially depend on  $x$ . We apply the Duhamel–Neumann law to equilibrium thermomechanical (thermoelastic) constitute relations.

$$\begin{cases} \epsilon_{xx} = \frac{1}{E} \sigma_{xx}^e + \alpha(\theta - \theta_0) \\ \epsilon_{yy} = -\frac{\nu}{E} \sigma_{xx}^e + \alpha(\theta - \theta_0) \\ \epsilon_{zz} = -\frac{\nu}{E} \sigma_{xx}^e + \alpha(\theta - \theta_0) \end{cases} \tag{50}$$

Therefore

$$\epsilon = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = (1 - 2\nu)\epsilon_{xx} + 2\alpha(1 + \nu)(\theta - \theta_0) \tag{51}$$

The governing equations becomes

$$\begin{cases} \rho \ddot{\epsilon}_{xx} = E \frac{\partial^2 \epsilon_{xx}}{\partial x^2} - E\alpha \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 \sigma_{xx}^{ne}}{\partial x^2} \\ \tau_\sigma \dot{\sigma}_{xx}^{ne} = k^\sigma \frac{\partial q}{\partial x} + 2\eta \dot{\epsilon}_{xx} - \sigma_{xx}^{ne} \\ \tau_q \dot{q} = -k \frac{\partial T}{\partial x} + k^\sigma \frac{\partial \sigma_{xx}^{ne}}{\partial x} - q \\ \left[ \rho c_v + 2\alpha\beta(1 + \nu)\theta - \frac{\tau_q}{k\theta^2} q^2 \right] \dot{T} = -\frac{\partial q}{\partial x} + \sigma_{xx}^{ne} \dot{\epsilon}_{xx} - E\alpha\theta \dot{\epsilon}_{xx} - \frac{\tau_\sigma}{2\eta} \sigma_{xx}^{ne} \dot{\sigma}_{xx}^{ne} - \frac{2\tau_q}{k\theta} q\dot{q}, \quad \text{where } T = \theta - \theta_0 \end{cases} \tag{52}$$

where Eq. (52a) is obtained by taking differential of Eq. (2) with respect to  $x$ . The thermal conduction coefficient  $k^\sigma = k_1^\sigma + k_2^\sigma + k_3^\sigma$ . In the following, we investigate the dissipation characteristics of the rod by studying its free and forced vibrations. Since the energy conservation equation in (52d) is nonlinear, for solutions to free vibrations, we used mode shapes of the corresponding nondissipative system to reduce a set of nonlinear partial differential equations (PDE) in (52) to a set of nonlinear ordinary differential equations (ODE) by Galerkin’s method. Then, a perturbation technique is used to solve nonlinear ODEs. For solutions to forced vibrations, we utilize a harmonic balance method. In these studies, we consider two types of thermomechanical boundary conditions: one is of fixed–fixed adiabatic ends, the other of free–free isothermal ends. In reality, the thermal boundary condition is an intermediate between adiabatic and isothermal conditions; the mechanical boundary condition is an intermediate of free-end and fixed-end conditions.

### 3.2. Dissipation characteristics of free vibrations

(1) Fixed–fixed adiabatic ends: the boundary conditions are

$$u(0, t) = u(L, t) = 0, \quad q(0, t) = q(L, t) = 0 \tag{53}$$

Besides these conventional boundary conditions, we need to address appropriate boundary conditions for temperature and nonequilibrium stress. According to Eq. (52b), we assume that the gradient of  $T$  is zero at boundaries. In this study,  $\sigma_{xx}^{ne}$  is set to zero at both ends. Displacement mode shapes for the corresponding nondissipative system are

$$\Phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots \tag{54}$$

Accordingly, mode shapes for  $\theta$ ,  $q_i$ , and  $\sigma_{xx}^{ne}$  can be properly addressed.

$$\begin{cases} \epsilon_{xx}(x, t) = \epsilon_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ \sigma_{xx}^{ne}(x, t) = \sigma_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ T(x, t) = T_n(t) \cos\left(\frac{n\pi x}{L}\right) \\ q(x, t) = q_n(t) \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots \end{cases} \tag{55}$$

By using the Galerkin’s method, we substitute Eq. (55) into (52a)–(52d), multiply Eqs. (52a), (52c) and (52d) by  $\cos(n\pi x/L)$ , (52b) by  $\sin(n\pi x/L)$ , and integrate them from 0 to  $L$ . Then, we can obtain the following nonlinear ODEs for the  $n$ th mode, up to cubic-order nonlinearity.

$$\begin{cases} \rho \ddot{\epsilon}_n = -\left(\frac{n\pi}{L}\right)^2 (E\epsilon_n - E\alpha T_n + \sigma_n) \\ \tau_\sigma \dot{\sigma}_n = k^\sigma \left(\frac{n\pi}{L}\right) q_n + 2\eta \dot{\epsilon}_n - \sigma_n \\ \tau_q \dot{q}_n = k \left(\frac{n\pi}{L}\right) T_n - k^\sigma \left(\frac{n\pi}{L}\right) \sigma_n - q_n \\ [\rho c_v + 2\alpha\beta(1 + \nu)\theta_0] \dot{T}_n = -\left(\frac{n\pi}{L}\right) q_n - E\alpha\theta_0 \dot{\epsilon}_n + \frac{\tau_q}{4k\theta_0^2} (q_n^2 \dot{T}_n + 2T_n q_n \dot{q}_n), \quad n = 1, 2, 3, \dots \end{cases} \tag{56}$$

3.2.1. Perturbation method

To solve for nonlinear ODEs in (56), we choose a small parameter  $\delta$  that measures magnitude of the deviation from a trivial state. The perturbation approach is justified. The bifurcating solution lies on branches of the form

$$\Phi_n(t) = \delta \Phi_n^1(t) + \delta^2 \Phi_n^2(t) + \delta^3 \Phi_n^3(t) + O(\delta^4), \quad n = 1, 2, 3, \dots \tag{57}$$

where  $\Phi$  denotes symbols ‘ $\epsilon$ ’, ‘ $\sigma$ ’, ‘ $q$ ’ and ‘ $T$ ’. If we denote a vector  $\{X\} = \{\epsilon_n, \sigma_n, q_n, T_n\}^T$ , then  $\{X\}$  can be written in terms of the powers of  $\delta$  as  $\{X\} = \delta\{X_1\} + \delta^2\{X_2\} + \delta^3\{X_3\}$ , where  $\{X_1\} = \{\epsilon_n^1, \sigma_n^1, q_n^1, T_n^1\}^T$ ,  $\{X_2\} = \{\epsilon_n^2, \sigma_n^2, q_n^2, T_n^2\}^T$ ,  $\{X_3\} = \{\epsilon_n^3, \sigma_n^3, q_n^3, T_n^3\}^T$ . Then, we can rewrite Eq. (56) in the order of powers of  $\delta$  as follows.

$\delta^1$ -order analysis

The  $\delta^1$ -order governing equations for the  $n$ th mode can be written in the form of state space.

$$[A]\{\dot{X}_1\} + [B]\{X_1\} = \{0\} \tag{58}$$

where

$$[A] = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \tau_\sigma & 0 & 0 \\ 0 & 0 & 0 & \tau_q & 0 \\ 0 & 0 & 0 & 0 & (\rho c_v + 2\beta\alpha(1 + \nu)\theta_0) \end{bmatrix}; \quad [B] = \begin{bmatrix} 0 & E\left(\frac{n\pi}{L}\right)^2 & \left(\frac{n\pi}{L}\right)^2 & 0 & -E\alpha\left(\frac{n\pi}{L}\right)^2 \\ -1 & 0 & 0 & 0 & 0 \\ -2\eta & 0 & 1 & -k^\sigma\left(\frac{n\pi}{L}\right) & 0 \\ 0 & 0 & k^\sigma\left(\frac{n\pi}{L}\right) & 1 & -k\left(\frac{n\pi}{L}\right) \\ E\alpha\theta_0 & 0 & 0 & \left(\frac{n\pi}{L}\right) & 0 \end{bmatrix} \tag{59}$$

This corresponds to an eigen-value problem.

$$(\lambda[A] + [B])\{X_1\} = \{0\} \tag{60}$$

For each  $n$ , there exist five roots. Three of them are negative, while the other two roots are complex conjugate with a negative real part, i.e.,  $\lambda = \xi_1 \pm i\xi_2$  (where  $\xi_1, \xi_2 \in \mathfrak{R}$ ;  $\xi_1 < 0$ ), which determines the modal information. The former three roots correspond to three relaxed motions, while the latter two the relaxed vibrations. For the  $n$ th mode, we denote its natural frequency as  $\omega_n$  and its damping ratio as  $\psi_n$ . We can determine  $\omega_n$  and  $\psi_n$  from the complex conjugates. The imaginary part of complex roots is equal to the damped natural frequency, i.e.,  $\omega_n = \xi_2$ ; the ratio of the absolute value of the real part to magnitude of the root gives the damping ratio, i.e.,  $\psi_n = |\xi_1|/|\lambda|$ . If corresponding normalized eigenvector is  $\{Z\} = \{(a_1 + ia_2), (b_1 + ib_2), (c_1 + ic_2), (d_1 + id_2), (e_1 + ie_2)\}^T$ , (where  $a_i, b_i, c_i, d_i, e_i \in \mathfrak{R}$ ), we can write the solution of Eq. (58).

$$\{X_1\} = \text{Re}(e^{\lambda t}\{Z\}); \quad \text{where } \lambda = \xi_1 + i\xi_2; \quad \xi_1, \xi_2 \in \mathfrak{R}; \quad \xi_1 < 0 \tag{61}$$

$\delta^2$ -order analysis

Similarly, the  $\delta^2$ -order governing equations can be written in the form of a state space. Same equations as for the  $\delta^1$ -order analysis are obtained for the  $\delta^2$ -order analysis.

$$[A]\{\dot{X}_2\} + [B]\{X_2\} = 0 \tag{62}$$

Therefore, the solution  $\{X_2\}$  is expected to be  $\{v X_1\}$  ( $v \in C$ ). However, the characteristic of  $\delta$  imposes  $v = 0$ . Then

$$\{X_2\} = 0 \tag{63}$$

$\delta^3$ -order analysis

Similarly, the  $\delta^3$ -order governing equations can also be written in a form of the state space.

$$[A]\{\dot{X}_3\} + [B]\{X_3\} = e^{(3\zeta_1+i\zeta_2)t}\{D_3\} + e^{3(\zeta_1+i\zeta_2)t}\{E_3\} \tag{64}$$

where

$$\begin{aligned} \{D_3\} &= \left(\frac{\tau_q \omega_n}{4k\theta_0^2}\right) \left\{ 0, 0, 0, 0, \left(-\frac{1}{4}(d_1^2 + 3d_2^2)e_2 - \frac{1}{2}d_1d_2e_1\right) + i\left(\frac{1}{4}(3d_1^2 + d_2^2)e_1 + \frac{1}{2}d_1d_2e_2\right) \right\}^T \\ \{E_3\} &= \left(\frac{\tau_q \omega_n}{4k\theta_0^2}\right) \left\{ 0, 0, 0, 0, \left(\frac{3}{4}(-d_1^2 + d_2^2)e_2 - \frac{3}{2}d_1d_2e_1\right) + i\left(\frac{3}{4}(d_1^2 - d_2^2)e_1 - \frac{3}{2}d_1d_2e_2\right) \right\}^T \end{aligned}$$

Two particular solutions correspond to the two types of excitations on the right-hand side of Eq. (64). They are

$$\begin{aligned} \{X_3\}_{p1} &= \text{Re}(e^{(3\zeta_1+i\zeta_2)t}[(3\zeta_1 + i\zeta_2)[A] + [B]]^{-1}\{D_3\}) \\ \{X_3\}_{p2} &= \text{Re}(e^{3(\zeta_1+i\zeta_2)t}[3(\zeta_1 + i\zeta_2)[A] + [B]]^{-1}\{E_3\}) \end{aligned} \tag{65}$$

Similarly, the general solution for the  $\delta^3$ -order is set to zero. Then the  $\delta^3$ -order solution is

$$\{X_3\} = \{X_3\}_{p1} + \{X_3\}_{p2} \tag{66}$$

Finally, from Eqs. (61), (63), and (66), we obtain an asymptotically approximated time function for the  $n$ th modal function.

$$\begin{aligned} \{X\} &= \delta\{X_1\} + \delta^3(\{X_3\}_{p1} + \{X_3\}_{p2}) \\ &= \text{Re}(\delta e^{\xi_1 t}(e^{i\xi_2 t}\{Z\})) + \delta^3 e^{3\xi_1 t}(e^{i\xi_2 t}((3\zeta_1 + i\zeta_2)[A] + [B])^{-1}\{D_3\} + e^{i3\xi_2 t}(3(\zeta_1 + i\zeta_2)[A] + [B])^{-1}\{E_3\})) \end{aligned} \tag{67}$$

From Eq. (67), we can see that the  $\delta^3$ -order component is damped three times faster than the  $\delta^1$ -order component. The  $\delta^3$ -order components are usually negligible. Therefore, for free vibrations, linear (or  $\delta^1$ -order) analysis is accurate enough.

(2) Free-free isothermal ends: the boundary conditions are given by

$$\sigma_{xx}^e(0, t) = \sigma_{xx}^{ne}(0, t) = T(0, t) = \sigma_{xx}^e(L, t) = \sigma_{xx}^{ne}(L, t) = T(L, t) = 0$$

Accordingly, we assume the following mode shapes.

$$\begin{cases} \varepsilon_{xx}(x, t) = \varepsilon_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ \sigma_{xx}^{ne}(x, t) = \sigma_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ T(x, t) = T_n(t) \sin\left(\frac{n\pi x}{L}\right) \\ q(x, t) = q_n(t) \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, 3, \dots \end{cases} \tag{68}$$

Where we set the gradient of heat flux zero at the two ends. The governing equations can be derived.

$$\begin{cases} \rho \ddot{\varepsilon}_n = -\left(\frac{n\pi}{L}\right)^2 (E\varepsilon_n - E\alpha T_n + \sigma_n) \\ \tau_\sigma \dot{\sigma}_n = -k^\sigma \left(\frac{n\pi}{L}\right) q_n + 2\eta \dot{\varepsilon}_n - \sigma_n \\ \tau_q \dot{q}_n = -k \left(\frac{n\pi}{L}\right) T_n + k^\sigma \left(\frac{n\pi}{L}\right) \sigma_n - q_n \\ \left(\frac{\rho c_v + 2\alpha\beta(1+\nu)\theta_0}{2\alpha\beta(1+\nu)\theta_0}\right) \dot{T}_n = \left(\frac{n\pi}{L}\right) q_n - E\alpha\theta_0 \dot{\varepsilon}_n + \frac{2}{3\pi} \left( \frac{\sigma_n \dot{\varepsilon}_n - E\alpha T_n \dot{\varepsilon}_n - \frac{\tau_\sigma}{2\eta} \sigma_n \dot{\sigma}_n -}{2\alpha\beta(1+\nu)\theta_0 T_n \dot{T}_n - \frac{\tau_q}{k\theta_0} q_n \dot{q}_n} \right) \\ \quad + \frac{\tau_q}{4k\theta_0^2} \left( \frac{q_n^2 \dot{T}_n +}{2T_n q_n \dot{q}_n} \right), \quad n = 1, 2, 3, \dots \end{cases} \tag{69}$$



Similarly, using the perturbation method, we can obtain asymptotic approximation for the  $n$ th modal function.

$$\{X\} = \delta\{X_1\} + \delta^2(\{X_2\}_{p1} + \{X_2\}_{p2}) + \delta^3(\{X_3\}_{p1} + \{X_3\}_{p2}) \tag{70}$$

where

$$\begin{aligned} \{X_1\} &= \text{Re}(e^{(\xi_1+i\xi_2)t}\{Z\}) \\ \{X_2\}_{p1} &= \text{Re}(e^{2\xi_1 t}[2\xi_1[A] + [B]]^{-1}\{D_2\}) \\ \{X_2\}_{p2} &= \text{Re}(e^{2(\xi_1+i\xi_2)t}[2(\xi_1 + i\xi_2)[A] + [B]]^{-1}\{E_2\}) \\ \{X_3\}_{p1} &= \text{Re}(e^{3\xi_1+i\xi_2 t}[(3\xi_1 + i\xi_2)[A] + [B]]^{-1}\{D_3\}) \\ \{X_3\}_{p2} &= \text{Re}(e^{3(\xi_1+i\xi_2)t}[3(\xi_1 + i\xi_2)[A] + [B]]^{-1}\{E_3\}) \end{aligned}$$

For free–free isothermal-end case, we obtain the same  $[A]$  and similar  $[B]$  as for fixed–fixed adiabatic-end case.  $\{Z\}$  is the corresponding eigenvector.  $\{D_3\}$  and  $\{E_3\}$  are defined in Eq. (64). The  $\delta^2$ -order excitations are given by  $e^{2\xi_1 t}\{D_2\}$  and  $e^{2(\xi_1+i\xi_2)t}\{E_2\}$ , and we have

$$\begin{aligned} \{D_2\} &= \left(\frac{\omega_n}{3\pi}\right)\{0, 0, 0, 0, [(-c_1b_2 + c_2b_1) - E\alpha(-e_1b_2 + e_2b_1)]\}^T \\ \{E_2\} &= \left(\frac{\omega_n}{3\pi}\right)\left\{0, 0, 0, 0, \left(\begin{aligned} &(-c_1b_2 - c_2b_1) - E\alpha(-e_1b_2 - e_2b_1) + \\ &\frac{\tau_q}{\eta}c_1c_2 + 4\alpha\beta(1 + \nu)\theta_0e_1e_2 + \frac{2\tau_q}{k\theta_0}d_1d_2 \end{aligned}\right) + i\left(\begin{aligned} &(c_1b_1 - c_2b_2) - E\alpha(e_1b_1 - e_2b_2) - \frac{\tau_q}{2\eta}(c_1^2 - c_2^2) - \\ &2\alpha\beta(1 + \nu)\theta_0(e_1^2 - e_2^2) - \frac{\tau_q}{k\theta_0}(d_1^2 - d_2^2) \end{aligned}\right)\right\}^T \end{aligned}$$

In the following, we studied the free vibrations of an aluminum rod subjected to either fixed–fixed adiabatic ends or free–free isothermal ends. The material constants for aluminum in International System Units are given in Table 1.

The rod is of length 10 m. Besides material constants given in the Table 1, material parameters introduced in EIT model are chosen as,  $\tau_\sigma = 1.0 \times 10^{-4}$  s,  $\tau_q = 1.0 \times 10^{-20}$  s,  $\eta = 3.5 \times 10^3$  N s/m<sup>2</sup>, and  $k^\sigma = 5.0 \times 10^{-3}$ . We chose the values of relaxation times,  $\tau_\sigma$  and  $\tau_q$ , from the range obtained by available experimental data discussed in the introduction. The comparison of damped and undamped modal parameters is presented in the Table 2. It is seen that the damped natural frequencies slightly deviate from the undamped natural frequencies, by about 0.2% higher or lower for the first 10 modes. The variation of damping ratio with natural frequency is shown in Fig. 3a, where nonequilibrium parameters are selected to produce a peak damping ratio at the 6th natural frequency. A nonlinear dependency on the frequency is observed for the damping ratio. By choosing the 6th mode, we studied the effects of nonequilibrium material parameters on damping ratio. The numerical results are presented in the Fig. 3b–d. In each plot, by fix other material parameters as listed above, only one parameter is chosen to vary. In Fig. 3b, where we chose the relaxation time  $\tau_q$  as the varying parameter, it is observed that although  $\tau_q$  is a key feature to determine the propagation speed of thermal disturbance,  $\tau_q$  does not affect the damping ratio much. As  $\tau_q$  increases from  $10^{-20}$  to  $10^{-5}$  s, the changes of damping ratio

Table 1  
The material properties

$\rho$ (kg/m <sup>3</sup> )	$2.7 \times 10^3$	$E$ (GPa)	$70 \times 10^9$	$\nu$	0.3
$\alpha$ (/k)	$23.6 \times 10^{-6}$	$k$ (J/s m k)	222	$c_v$ (J/kg k)	900

Table 2  
Comparison of undamped and damped modal parameters ( $\omega_2$ , damped frequency;  $\omega_1$ , undamped frequency;  $\Delta\omega = \omega_2 - \omega_1$ )

Mode No.	$\omega_1$ (rad/s)	Free–free isothermal-end case			Fixed–fixed adiabatic-end case		
		$\omega_2$ (rad/s)	$\Delta\omega/\omega_1$ (%)	$\psi_n$ ( $10^{-3}$ )	$\omega_2$ (rad/s)	$\Delta\omega/\omega_1$ (%)	$\psi_n$ ( $10^{-3}$ )
1	1599.62	1595.90	−0.23	0.0781	1603.14	0.22	0.0778
6	9597.72	9577.60	−0.21	0.2509	9622.40	0.25	0.2486
10	15996.21	15964.59	−0.20	0.2259	16039.24	0.26	0.2234

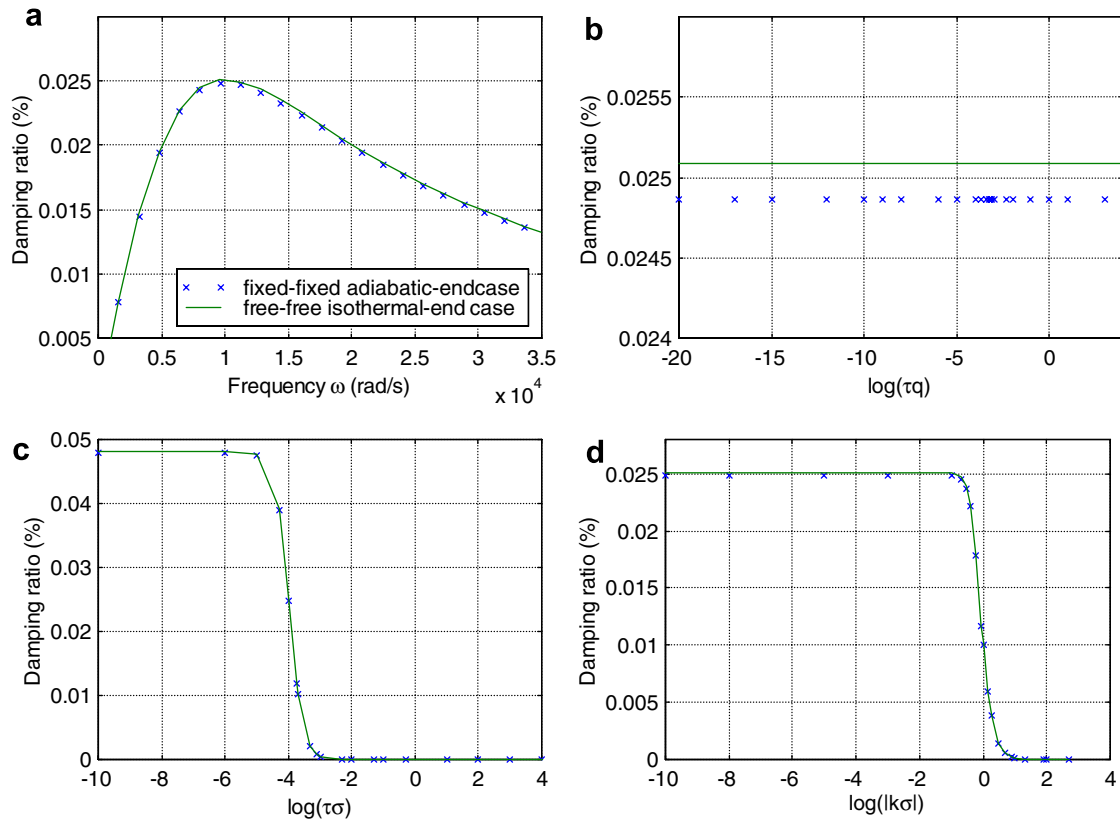


Fig. 3. The variations of damping ratio with: (a) natural frequency  $\omega$ ; (b) relaxation time  $\tau_q$ ; (c) relaxation time  $\tau_\sigma$ ; (d) thermal-viscous constant  $k_\sigma$ .

are negligible. On the contrary, the other two parameters  $\tau_\sigma$  and  $k^\sigma$  take more important roles in affecting the damping ratio. As shown in the Fig. 3c, where we chose the relaxation time  $\tau_\sigma$  as the varying parameter, damping ratio increases quickly from zero, as  $\tau_\sigma$  decreases from  $10^{-3}$  to  $10^{-5}$  s. However, as  $\tau_\sigma$  is larger than  $10^{-3}$  s or less than  $10^{-5}$  s, damping ratio changes very little. As is stated, thermal-viscous constant  $k^\sigma$  reflects the coupling between nonequilibrium thermal and viscous fields. Large value of  $k^\sigma$  means a strong coupling. As in the Fig. 3d, where  $k^\sigma$  is the varying parameter, the variation of damping ratio is quite large with respect to  $k^\sigma$  in the range of  $10^{-1}$  to 10. It is observed that a strong coupling reduces the damping ratio, while a weak coupling increases it. In this particular case, damping ratio can drop from 0.025% to almost zero as the coupling coefficient increases from  $10^{-1}$  to 10. Based on these observations, we can conclude that nonequilibrium parameters  $\tau_\sigma$  and  $k^\sigma$  can strongly affect the damping characteristics. To make things more clearly, we selected various values of  $\tau_\sigma$  and  $k^\sigma$  and compared the associated variation of damping ratio versus frequency. Results are shown in Fig. 4. In Fig. 4a, we fixed  $\tau_\sigma = 10^{-4}$  s and varied the coupling coefficient  $k^\sigma$ ; while in Fig. 4b, we fixed  $k^\sigma = 5 \times 10^{-3}$  and varied the relaxation time  $\tau_\sigma$ . We can see that these two parameters not only change the magnitude of damping ratio but also the profile of damping ratio with respect to frequency, including the shifting of corresponding frequency of maximum damping ratio. As  $\tau_\sigma \rightarrow 0$ , a nonlinear dependency of damping ratio on frequency tends to a linear dependency (Fig. 4a and Fig. 8a). In this particular case, either increasing relaxation time  $\tau_\sigma$  or increasing coupling coefficient  $k^\sigma$  reduces the damping ratio.

In Fig. 5, we calculate the 6th modal function for free–free isothermal-end case as in Eq. (70). We select  $\delta = 1$  and choose the maximum value of  $\delta^1$  component as a reference strain to normalize. The normalized  $\delta^1$ ,  $\delta^2$ , and  $\delta^3$  order components for  $\varepsilon_{xx}(x,t)$  at  $x = L/8$  are presented in Fig. 5a–c. As is expected, it is observed that  $\varepsilon^2$  and  $\varepsilon^3$  order components are quite smaller than  $\varepsilon^1$  component and are negligible.

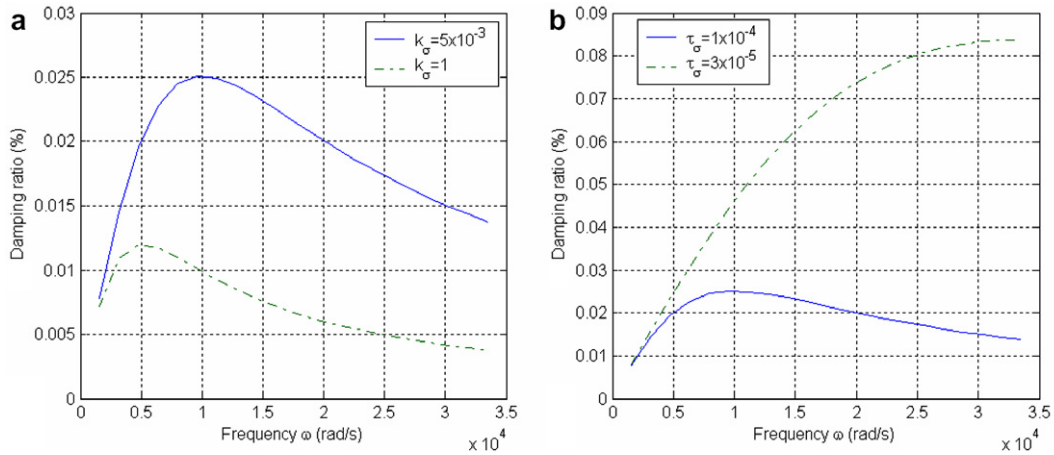


Fig. 4. Damping ratio versus frequency: (a) at selected  $k^\sigma$  and a fixed  $\tau_\sigma = 10^{-4}$  s; (b) at selected  $\tau_\sigma$  and a fixed  $k^\sigma = 5 \times 10^{-3}$ .

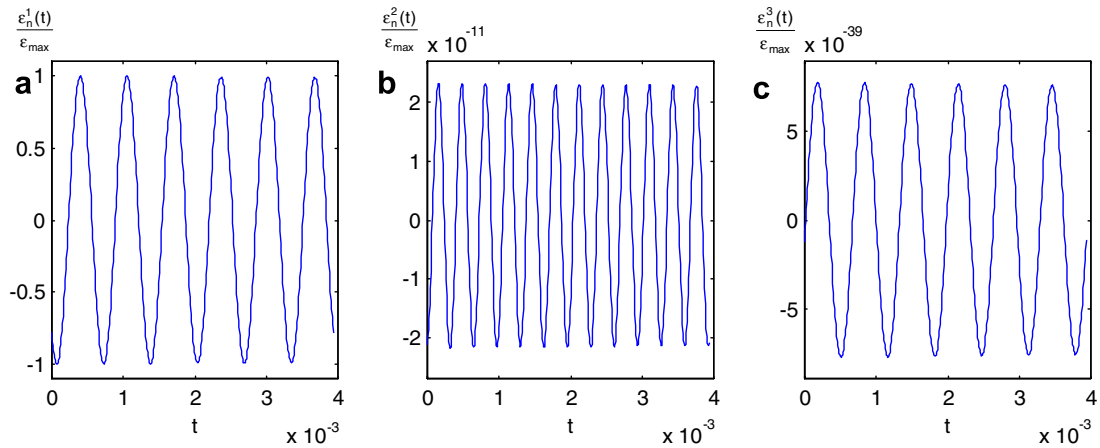


Fig. 5. The 6th modal function at  $x = L/8$  for free-free isothermal-end case. (a)  $\delta^1$  order component; (b)  $\delta^2$  order component; (c)  $\delta^3$  order component (where  $\epsilon_{\max} = \max(\epsilon_n^i(t))$ ).

### 3.3. Energy dissipation of forced vibrations

A uniformly distributed harmonic longitudinal strain is applied to study the dissipation of the rod during its cyclic forced vibrations.

$$\epsilon_{xx} = \epsilon_0 \text{Re}\{e^{i\omega t}\} \tag{71}$$

Since the energy balance equation in (53d) is nonlinear, variables  $\sigma_{xx}^{ne}$ ,  $T$ , and  $q$  would experience multi-harmonic vibrations. In this work, we solve for these motions up to second-order harmonics. Then, we can assume

$$\begin{cases} \sigma_{xx}^{ne}(x, t) = \text{Re}\{\hat{\sigma}_1(x)e^{i\omega t} + \hat{\sigma}_2(x)e^{i2\omega t}\} \\ T(x, t) = \text{Re}\{\hat{T}_1(x)e^{i\omega t} + \hat{T}_2(x)e^{i2\omega t}\} \\ q(x, t) = \text{Re}\{\hat{q}_1(x)e^{i\omega t} + \hat{q}_2(x)e^{i2\omega t}\} \end{cases} \tag{72}$$

According to the harmonic balance method, we substitute Eqs. (71) and (72) into (53), neglect the super harmonic terms higher than second-order harmonics, and collect terms at time scales  $e^{i\omega t}$  and  $e^{i2\omega t}$ , respectively. Then, we obtain

$$\begin{cases} \hat{\sigma}_1 = (1 + i\omega\tau_\sigma)^{-1} (k^\sigma \frac{d\hat{q}_1}{dx} + 2i\omega\eta\varepsilon_0) \\ \hat{q}_1 = (1 + i\omega\tau_q)^{-1} (-k \frac{d\hat{T}_1}{dx} + k^\sigma \frac{d\hat{\sigma}_1}{dx}) \\ \hat{T}_1 = [i\omega(\rho c_v + 2\alpha\beta(1 + \nu)\theta_0)]^{-1} (-\frac{d\hat{q}_1}{dx} - i\omega E\alpha\theta_0\varepsilon_0) \end{cases} \quad (73)$$

and

$$\begin{cases} \hat{\sigma}_2 = (1 + i2\omega\tau_\sigma)^{-1} (k^\sigma \frac{d\hat{q}_2}{dx}) \\ \hat{q}_2 = (1 + i2\omega\tau_q)^{-1} (-k \frac{d\hat{T}_2}{dx} + k^\sigma \frac{d\hat{\sigma}_2}{dx}) \\ \hat{T}_2 = [i2\omega(\rho c_v + 2\alpha\beta(1 + \nu)\theta_0)]^{-1} [-\frac{d\hat{q}_2}{dx} + i\frac{1}{2}\omega\varepsilon_0(\hat{\sigma}_1 - E\alpha\hat{T}_1) \\ -i\omega\alpha\beta(1 + \nu)(\hat{T}_1)^2 - i\omega \frac{\tau_\sigma}{4\eta} (\hat{\sigma}_1)^2 - i\omega \frac{\tau_q}{k\theta_0} (\hat{q}_1)^2] \end{cases} \quad (74)$$

To solve for Eq. (73), substituting Eqs. (73a) and (73c) into (73b), we obtain a governing equation for  $\hat{q}_1$ .

$$\frac{d^2\hat{q}_1}{dx^2} - \frac{(1 + i\omega\tau_q)}{(A1 + iA2)}\hat{q}_1 = 0 \quad (75)$$

where  $A1 = \frac{(k^\sigma)^2}{1 + (\omega\tau_\sigma)^2}$ ;  $A2 = -(\omega\tau_\sigma A1 + \frac{k}{\omega A})$ ;  $A = \rho c_v + 2\alpha\beta(1 + \nu)\theta_0$

We solve for Eq. (75), then substitute the solution into Eqs. (73a) and (73c) to solve for  $\hat{\sigma}_1(x)$  and  $\hat{T}_1(x)$ . We obtain the following solutions.

$$\begin{cases} \hat{q}_1(x) = \varepsilon_0 [C \exp((\xi_1 + i\xi_2)x) + D \exp(-(\xi_1 + i\xi_2)x)] \\ \hat{\sigma}_1(x) = \varepsilon_0 [\hat{\sigma}_1^1(x) + \hat{\sigma}_1^2] \\ \hat{T}_1(x) = \varepsilon_0 [\hat{T}_1^1(x) + \hat{T}_1^2] \end{cases} \quad (76)$$

where  $C$  and  $D$  are complex constants to be determined by thermal boundary conditions, and

$$\begin{aligned} \xi_1 &= r \cos(\phi); \quad \xi_2 = r \sin(\phi); \quad r = \frac{|1 + i\omega\tau_q|^{\frac{1}{2}}}{|A1 + iA2|}; \quad \phi = \frac{1}{2} \left[ \arctan(\omega\tau_q) - \arctan\left(\frac{A2}{A1}\right) \right] \\ \hat{\sigma}_1^1(x) &= (1 + i\omega\tau_\sigma)^{-1} \{k^\sigma (\xi_1 + i\xi_2) [C \exp((\xi_1 + i\xi_2)x) - D \exp(-(\xi_1 + i\xi_2)x)]\} \\ \hat{\sigma}_1^2 &= i2\eta\omega(1 + i\omega\tau_\sigma)^{-1} \\ \hat{T}_1^1(x) &= (i\omega A)^{-1} \{-(\xi_1 + i\xi_2) [C \exp((\xi_1 + i\xi_2)x) - D \exp(-(\xi_1 + i\xi_2)x)]\} \\ \hat{T}_1^2 &= -\frac{E\alpha\theta_0}{A} \end{aligned}$$

To solve Eq. (74), substituting Eqs. (74a) and (74c) into (74b), we can obtain the governing equation for  $\hat{q}_2$ .

$$\begin{aligned} \frac{d^2\hat{q}_2}{dx^2} - \frac{(1 + i2\omega\tau_q)}{(B1 + iB2)}\hat{q}_2 \\ = \frac{k\varepsilon_0}{2A(B1 + iB2)} \left\{ \left( 1 - \frac{\tau_\sigma\varepsilon_0}{2\eta} \hat{\sigma}_1^2 \right) \frac{d\hat{\sigma}_1^1}{dx} - (E\alpha + \alpha\beta(1 + \nu)\varepsilon_0\hat{T}_1^2) \frac{d\hat{T}_1^1}{dx} \right. \\ \left. - \varepsilon_0 \left( \alpha\beta(1 + \nu) \frac{d}{dx} (\hat{T}_1^1)^2 + \frac{\tau_\sigma}{4\eta} \frac{d}{dx} (\hat{\sigma}_1^1)^2 + \frac{\tau_q}{k\theta_0} \frac{d}{dx} (\hat{q}_1)^2 \right) \right\} \quad (77) \end{aligned}$$

where  $B1 = \frac{(k^\sigma)^2}{1 + (2\omega\tau_\sigma)^2}$ ;  $B2 = -(2\omega\tau_\sigma B1 + \frac{k}{2\omega A})$

The solution for Eq. (77) is given by

$$\hat{q}_2(x) = \varepsilon_0 [C_1 \exp((\zeta_1 + i\zeta_2)x) + D_1 \exp(-(\zeta_1 + i\zeta_2)x)] + \frac{k\varepsilon_0}{2A(B1 + iB2)[(\zeta_1 + i\zeta_2)^2 - (\zeta_1 + i\zeta_2)^2]} \left[ \left( 1 - \frac{\tau_\sigma \varepsilon_0}{2\eta} \hat{\sigma}_1^2 \right) \frac{d\hat{\sigma}_1^1}{dx} - (E\alpha + \alpha\beta(1 + \nu)\varepsilon_0 \hat{T}_1^2) \frac{d\hat{T}_1^1}{dx} \right] - \frac{k\varepsilon_0^2}{2A(B1 + iB2)[4(\zeta_1 + i\zeta_2)^2 - (\zeta_1 + i\zeta_2)^2]} \left( \alpha\beta(1 + \nu) \frac{d}{dx} (\hat{T}_1^1)^2 + \frac{\tau_\sigma}{4\eta} \frac{d}{dx} (\hat{\sigma}_1^1)^2 + \frac{\tau_q}{k\theta_0} \frac{d}{dx} (\hat{q}_1)^2 \right) \tag{78}$$

where  $C_1$  and  $D_1$  as complex constants to be determined by thermal boundary conditions, and

$$\zeta_1 = r_1 \cos(\phi_1); \quad \zeta_2 = r_1 \sin(\phi_1); \quad r_1 = \left| \frac{1 + i2\omega\tau_q}{B1 + iB2} \right|^{\frac{1}{2}}; \quad \phi_1 = \frac{1}{2} \left[ \arctan(2\omega\tau_q) - \arctan\left(\frac{B2}{B1}\right) \right]$$

Then we can obtain  $\hat{\sigma}_2(x)$  and  $\hat{T}_2(x)$  by substituting Eq. (78) into (74a) and (74c). In overall, we have four complex constants  $C, D, C_1$  and  $D_1$  to be determined. For adiabatic boundary conditions where  $q(0) = q(L) = 0$ , it yields  $\hat{q}_i(0) = \hat{q}_i(L) = 0 (i = 1, 2)$ . We can obtain  $C = D = C_1 = D_1 = 0$ . For isothermal boundary conditions where  $T(0) = T(L) = 0$ , it yields  $\hat{T}_i(0) = \hat{T}_i(L) = 0 (i = 1, 2)$ . From it,  $C, D, C_1$  and  $D_1$  can be determined.

In the following numerical studies, we calculated the damping capacity defined in Eqs. (48)–(50) during a cycle of forced vibrations. The maximum elastic strain energy is given by

$$W = \max_{t \in [0, \frac{2\pi}{\omega}]} \rho e^e = \frac{1}{2} E \varepsilon_0^2 \tag{79}$$

The local nonequilibrium internal energy change per cycle is given by

$$\Delta W = \oint \rho \dot{e}^n dt = \oint (-q_{i,i} + \sigma_{ij} \dot{e}_{ij} - \rho \dot{e}^e) dt = \int_0^{\frac{2\pi}{\omega}} (-q_{i,i} + \sigma_{ij}^{ne} \dot{e}_{ij} + \beta \dot{\theta} \varepsilon - 3\alpha\beta T \dot{T}) dt \tag{80}$$

The variations of volume-averaged specific damping capacity  $\psi$ , defined in Eq. (49), with driving frequency  $\omega$  for both either isothermal or adiabatic ends, are studied. The results are shown in Fig. 6. We can see that the variation of volume-averaged specific damping ratio  $\psi$  with excitation frequency  $\omega$  is similar to the variation of damping ratio with natural frequency in free vibration as shown in Fig. 3a. However, when  $\omega$  is small, the  $\psi$  for adiabatic-boundary case is larger than that for isothermal boundary case; however  $\psi$  smaller, when  $\omega$  is

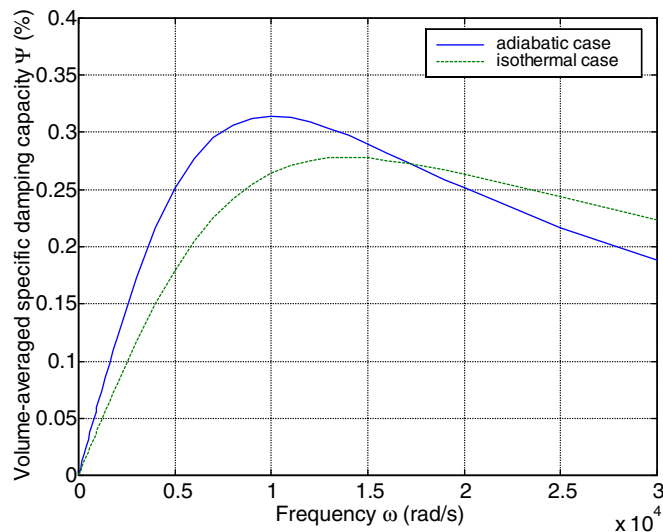


Fig. 6. The variation of volume-averaged specific damping capacity  $\psi$  with frequency  $\omega$ .

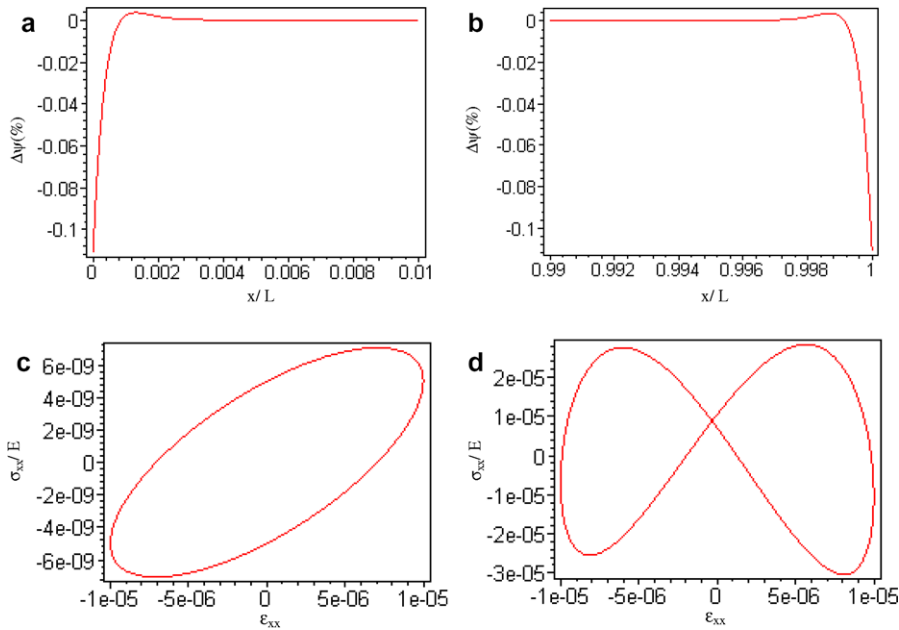


Fig. 7. The distribution of local specific damping capacity  $\Delta\psi$ : (a) along  $x/L$  from 0 to 0.01; (b) along  $x/L$  from 0.99 to 1. The nonequilibrium-stress and strain relations during a cycle: (c) at  $x/L = 0.01$  or  $0.99$ ; (d) at  $x/L = 0$  or  $1$ .

large. The distribution of local specific damping capacity  $\Delta\psi$  at  $\omega = 10^4$  rad/s (close to the 6th natural frequency) for the isothermal case is presented in Fig. 7. In this work, the local specific damping capacity  $\Delta\psi$  is defined as the ratio of the change of local nonequilibrium internal energy per cycle to the maximum strain energy. It includes two contributions: one is related to the local energy dissipation that is nonnegative, and the other to the flux of nonequilibrium internal energy. We can see in Fig. 7a and b that  $\Delta\psi$  as a function of  $x$  is not always positive. However, the volume-averaged specific damping ratio  $\psi$  is always positive (Fig. 6). Because when we integrate  $\Delta\psi$  along  $x$ , the flux of nonequilibrium internal energy cancels with each other and only the nonnegative dissipation part is left over. As is seen, the distribution of  $\Delta\psi$  gives us the information about the flow of nonequilibrium internal energy during a cycle. The nonequilibrium stress and strain relations at various locations are presented in Fig. 7c and d. At  $x/L = 0$  or  $1$ , the second harmonic component for nonequilibrium stress is large, then we get a butterfly style curve (Fig. 7d); at  $x/L = 0.01$  or  $0.99$ , the second harmonic component for nonequilibrium stress is quite smaller than the harmonic component, an elliptical curve is obtained (Fig. 7c). As  $x/L$  from 0.01 to 0.99, the nonequilibrium stress-strain curve does not change. It corresponds with what we observed in Fig. 7a and b, i.e.,  $\Delta\psi$  keeps unvarying almost immediately apart from the ends. Although the second harmonic component makes a significant contribution to the stress–strain relation curve, it does not contribute to the dissipation since it is orthogonal to the excitation.

#### 4. In comparison to other thermodynamics damping models

##### 4.1. Visco-thermoelastic model (CIT)

As mentioned in the introduction section, both linear viscous damping and thermoelastic damping are CIT models. We combine these two models to formulate a visco-thermoelastic model that is given in Eq. (43). The specific CIT internal energy is given by

$$\rho e = \frac{1}{2}\lambda(\epsilon_{kk})^2 + \mu\epsilon_{ij}\epsilon_{ij} + \beta\theta_0\epsilon_{kk} + \rho c_v(\theta - \theta_0) + \rho e_0 \tag{81}$$

For longitudinal vibrations of a uniform rod, the governing equations are

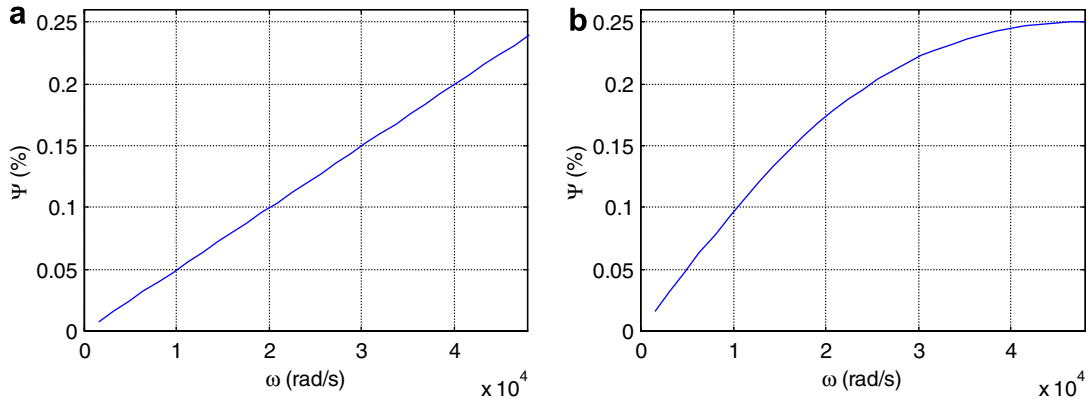


Fig. 8. The variation of damping ratio with frequency for the case of fixed–fixed adiabatic ends: (a) visco-thermoelastic model; (b) IVT model ( $n = 1$ ).

$$\begin{cases} \rho \ddot{\epsilon}_{xx} = E \frac{\partial^2 \epsilon_{xx}}{\partial x^2} - E\alpha \frac{\partial^2 T}{\partial x^2} + 2\eta \frac{\partial^2 \dot{\epsilon}_{xx}}{\partial x^2} \\ \rho c_v \dot{T} = k \frac{\partial^2 T}{\partial x^2} + 2\eta \dot{\epsilon}_{xx}^2 - E\alpha\theta \dot{\epsilon}_{xx} - 2\alpha\beta(1 + \nu)\theta \dot{T} \end{cases} \quad (82)$$

Following the same procedure as in the study of EIT model, we obtain the variation of the damping ratio with respect to natural frequency for the case of fixed–fixed adiabatic ends. Same material constants are used. The results are presented in Fig. 8a. In contrary to the EIT model, as mentioned early, we observe a linear dependency of damping ratio on the frequency, which is a limiting case of  $\tau_\sigma \rightarrow 0$ .

4.2. Internal variable theory model (RT)

A simple IVT model that introduces only one internal variable, i.e. an anelastic strain  $\epsilon_{ij}^{an}$ , is developed. The thermal field is considered. The anelastic strain follows a 1st order evolution. For longitudinal vibrations of a uniform rod, we have

$$\begin{cases} \sigma_{ij} = \lambda \epsilon \delta_{ij} + 2\mu \epsilon_{ij}^- \beta T \delta_{ij} \\ q_i = -k \frac{\partial T}{\partial x} \\ \tau_\epsilon \dot{\epsilon}_{ij}^{an} = -\alpha_1 \dot{\epsilon}_{ij}^{an} - \alpha_2 \epsilon_{ij}^e, \quad \text{where } \epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^{an} \end{cases} \quad (83)$$

where  $\tau_\epsilon, \alpha_1, \alpha_2 > 0$ . The internal variable, i.e., the anelastic strain, can be eliminated from Eq. (83c) through the following integration.

$$\epsilon_{ij}^{an} = -\frac{1}{\tau_\epsilon} \int_{-\infty}^t \alpha_1 \epsilon_{xx}^e \exp\left[-\frac{\alpha_2}{\tau_\epsilon}(t - \tau)\right] d\tau \quad (84)$$

With only one internal variable, specific internal energy can be uniquely determined.

$$\rho e = \left(1 - \frac{\alpha_2}{\alpha_1}\right) \left(\frac{1}{2} \lambda (\epsilon^e)^2 + \mu \epsilon_{ij}^e \epsilon_{ij}^e\right) + \frac{\alpha_2}{2\alpha_1} \lambda \left(\epsilon^e + \frac{\alpha_1}{\alpha_2} \epsilon^{an}\right)^2 + \frac{\alpha_2}{\alpha_1} \mu \left(\epsilon_{ij}^e + \frac{\alpha_1}{\alpha_2} \epsilon_{ij}^{an}\right)^2 + \beta T_{x0} \epsilon^e + \rho c_v (T_x - T_{x0}) + \rho e_0 \quad (85)$$

where  $T = T_x - T_{x0}$ ,  $\epsilon^e = \epsilon_{kk}^e$  and  $\epsilon^{an} = \epsilon_{kk}^{an}$ .  $T_x$  is the local accompanying absolute temperature, and  $T_{x0}$  is the reference temperature. The governing equations reduce to

$$\begin{cases} \rho \ddot{\epsilon}_{xx}^e = E \frac{\partial^2 \epsilon_{xx}^e}{\partial x^2} - E\alpha \frac{\partial^2 T}{\partial x^2} - \frac{E}{\tau_\epsilon} \int_{-\infty}^t \alpha_1 \frac{\partial^2 \epsilon_{xx}^e}{\partial x^2} \exp\left[-\frac{\alpha_2}{\tau_\epsilon}(t - \tau)\right] d\tau + X(x, t) \\ \rho c_v \dot{T} = k \frac{\partial^2 T}{\partial x^2} + E(1 - \frac{\alpha_2}{\alpha_1}) \dot{\epsilon}_{xx}^e \epsilon_{xx}^{an} - E\alpha(T + T_{x0}) \dot{\epsilon}_{xx}^e - E\alpha T \dot{\epsilon}_{xx}^{ne} \end{cases} \quad (86)$$

By assuming mode shapes, we apply a Galerkin's method for the solutions. For fixed–fixed adiabatic ends, we obtain a set of nonlinear integro-differential equations for the  $n$ th mode.

$$\begin{cases} \rho \ddot{\varepsilon}_n = -\left(\frac{n\pi}{L}\right)^2 (E\varepsilon_n - E\alpha T_n - \frac{E}{\tau_e} \int_{-\infty}^t \alpha_1 \varepsilon_n \exp\left[-\frac{\alpha_1}{\tau_e}(t-\tau)\right] d\tau) + X_n(t) \\ \rho c_v \dot{T}_n = -\left(\frac{n\pi}{L}\right)^2 k T_n - E\alpha T_{\alpha 0} \dot{\varepsilon}_n \end{cases} \quad (87)$$

By taking Laplace transformation, the transfer function can be obtained as follows

$$G_n(s) = \frac{X_n(s)}{\varepsilon_n(s)} = \left\{ \rho s^2 + E \left(\frac{n\pi}{L}\right)^2 \left( 1 - \frac{\alpha_2}{\alpha_1 + \tau_e s} + \frac{E\alpha^2 T_{\alpha 0} s}{\rho c_v s^2 + k \left(\frac{n\pi}{L}\right)^2} \right) \right\}^{-1} \quad (88)$$

By calculating the poles of transfer function for the  $n$ th mode, the variation of damping ratio with natural frequency is obtained. The results are shown in Fig. 8b. We have used the material constants in Table 1 and chosen  $\tau_e = 2.0 \times 10^{-5}$  s,  $\alpha_1 = 1.0$ , and  $\alpha_2 = 6.8 \times 10^{-3}$ . It is observed a nonlinear dependency of damping ratio on the natural frequency. As stated by Lesieutre et al. (Lesieutre and Mingori, 1990; Enelund and Lesieutre, 1999), IVT models are expected to have flexibility to present various nonlinear dependencies of damping ratio on frequency.

## 5. Conclusion

Material damping is a complicated thermodynamics phenomenon, which involves various irreversible thermodynamic processes and is associated with the interchange of various forms of energy. A hybrid thermodynamics framework is proposed for the development of general damping models for complex materials. In this paper, a damping model has been developed for a typical simple material in the framework of extended irreversible thermodynamics. For each involved irreversible process, a thermodynamic flux, which is driven by the corresponding thermodynamic forces, was introduced. Interferences among various irreversible processes are included in a first-order relaxation equation. Introduction of the relaxation characteristics for these thermodynamic fluxes describes their nonequilibrium characteristics and yields a propagation equation for either mechanical or thermal disturbance, and thus avoids a diffusion equation of infinite propagation speed. The model gives a physical insight on mechanisms of material damping by describing the irreversible interchanges of various forms of energy. The formulation includes a derivation for the nonlinear energy balance equation by using the first and second laws of thermodynamics. The reductions of the developed model to various thermodynamic damping models have been investigated by deducing various limiting cases.

By using the developed model, we have studied the vibrations of a long rod. A perturbation method and a harmonic balance method have been used to solve its free and forced vibrations, respectively. A nonlinear dependency of the damping ratio on frequency is studied and justified. Effects of nonequilibrium inter-field couplings and relaxation times are also investigated. Numerical studies show that these nonequilibrium parameters can significantly affect the dissipation characteristics in some situations.

In the future work, we will attempt to incorporate the internal variable theory into EIT to develop damping models for more general materials.

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