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NUMERICAL ANALYSIS OF LARGE-SCALE SOUND FIELDS USING ITERATIVE METHODS PART I: APPLICATION OF KRYLOV SUBSPACE METHODS TO BOUNDARY ELEMENT ANALYSIS*

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The convergence behavior of the Krylov subspace iterative solvers towards the systems with the 3D acoustical BEM is investigated through numerical experiments. The fast multipole BEM, which is an efficient BEM based on the fast multipole method, is used for solving problems with up to about 100,000 DOF. It is verified that the convergence behavior of solvers is much affected by the formulation of the BEM (singular, hypersingular, and Burton-Miller formulation), the complexity of the shape of the problem, and the sound absorption property of the boundaries. In BiCG-like solvers, GPBiCG and BiCGStab2 have more stable convergence than others, and these solvers are useful when solving interior problems in basic singular formulation. When solving exterior problems with greatly complex shape in Burton-Miller formulation, all solvers hardly converge without preconditioning, whereas the convergence behavior is much improved with ILU-type preconditioning.

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In these cases GMRes is the fastest, whereas CGS is one of the good choices, when taken into account the difficulty of determining the timing of restart for GMRes. As for calculation for rigid thin objects in hypersingular formulation, much more rapid convergence is observed than ordinary interior/exterior problems, especially using BiCG-like solvers.

Keywords: Large-scale sound field analysis; boundary element method; iterative method; Krylov subspace method; convergence; preconditioning; fast multipole method.

1. Introduction

1.1. *Background*

Iterative methods, especially Krylov subspace methods, have been widely known and used as rapid solvers for large systems of linear equations in recent years.[1,2] In the field of computational acoustics, these methods have been also applied to various analyses and proved to be powerful: FE[3,4] and BE[5] analyses of sound fields in rooms, BE analyses for diffuse and radiation problems,[6–8] coupling problems,[9] and noise propagation problems with three-dimensional large-scale noise barriers,[10] etc. Moreover, these methods have been employed for development of various efficient methods for analyzing large-scale problems, and the characteristic of the iterative methods have been used not only for improvement in the calculation speed but for the reduction of required memory of the efficient methods. We have also developed efficient methods for FE[3] and BE[5] analyses using Krylov subspace methods.

Iterative methods are techniques that use successive approximations to obtain more accurate solutions to a linear system at each step. The rate at which an iterative solver converges relates directly to its computational time. Regarding non-Hermitian matrices generally obtained in FEM and BEM, the convergence of iterative solvers is not clear, and depends greatly on the characteristics of problems.[11] Thus it is important to know the relation among the characteristics of problems, the convergence behavior and setting for solvers (about initial values, preconditioning, etc.), in order to prevents divergence of calculation due to unappropriate use of solvers and to reduce/estimate the computational time. The knowledge of the convergence behavior of iterative solvers is indispensable also for effective use of various efficient methods mentioned above.

1.2. *Purpose*

In our study, we make investigation for appropriate application of iterative methods to FE and BE analyses for sound fields. Especially the convergence behavior is investigated in detail. In the paper "Part I", we deal with the application of iterative methods to BE analyses.

Several researchers have studied on the convergence of iterative solvers for the BEM.[9,12] As for numerical experiments, however, many studies dealt with problems with simple shape objects that can be analytically solved or small-scale problems. Recently, Marburg *et al.* have investigated in detail the performance of four kinds of Krylov subspace solver (RBiCGStab, GMRes, CGNR and TFQMR) on some sound fields including three-dimensional practical fields such as a field in a small hall,[13] and concluded that GMRes (without restart) is the most robust solver. However, solvers based on Arnoldi process such as GMRes have a drawback that cannot be disregarded: the amount of work and memory required per iteration rises linearly with the iteration count, thus, one need to restart the process every l iterations to overcome this limitation (GMRes(l)). It is quite difficult to choose an appropriate value for l, since GMRes may converge slowly, or completely fail to converge if l is too small.[11] Moreover, Marburg *et al.* did not investigate the convergence of CGS and GPBiCG, well-known solvers based on Lanczos process, in which the amount of work and memory required per iteration does not increase with the iteration count. It is worth investigating the applicability of the solvers based on Lanczos process to the acoustic BEM in detail, since these solvers are better than GMRes in efficiency and in ease to handle if the convergence behavior is the same.

BE analyses for thin objects are often executed. In general thin-object problems, the spatial coincidence of the two sides of the thin object makes it impossible to solve the problems using only singular equations or only hypersingular equations; both singular and hypersingular equations are necessary (dual BEM).[14,15] However, in special cases where the thin objects are completely rigid, the sound field can be solved using only hypersingular equations, if the difference of sound pressure between two sides of the object is considered to be unknowns.[16] This technique is often used for analysis of noise barriers, sound diffusers, etc. To our knowledge, the convergence behavior for rigid thin objects using this technique has not been reported yet.

In the present paper, we investigate through numerical experiments the performance of Krylov subspace iterative solvers and the effect of methods for the improvement of convergence in three-dimensional acoustic BE analyses. We compare convergence behavior between BiCG-like solvers, which are a main group of solvers based on Lanczos process, and GMRes, which is known as a robust solver. Since we solve relatively large-scale problems with up to about 100,000 degrees of freedom (DOF) for the investigation, we use the fast multipole BEM (FMBEM), which is an efficient BEM based on the fast multipole method (FMM) originally proposed by Rokhlin.[17,18] The FMM[19–25] and the FMBEM[5–8,26–28] in the field of acoustics have been studied by a lot of researchers. For three-dimensional problems, the FMM based on the diagonal form by Rokhlin (FMM for high frequencies)[20] is widely applied because the original FMM takes a lot of computational costs in three-dimensional fields. However, it is well-known that the diagonal form of Rokhlin suffers from instability for small wave numbers. To resolve this problem, the new version of the FMM has been proposed (FMM for low frequencies),[24] and a wideband version of FMM in which the FMMs for high and low frequencies are unified has also been proposed.[25] In the present paper we use the FMBEM[5] based on the diagonal form by Rokhlin. For the detail review on the FMM, an intensive review paper by Nishimura[29] can be available.

Section 2 briefly discusses iterative solvers and methods for the improvement of convergence, used in this paper. In Sec. 3, we examine the performance of BiCG-like iterative solvers only, towards some interior and exterior problems including practical models. We investigate the effects of boundary conditions, boundary shapes, and kinds of method for improvement of convergence on the convergence behavior in three kinds of formulation, *i.e.* basic formulation (singular formulation: SF), normal derivative formulation (hypersingular formulation: HF), and formulation by Burton and Miller (BM)[30] to avoid fictitious eigenfrequency difficulties in exterior problems. In Sec. 4, we compare the BiCG-like solvers showing better convergence in Sec. 3 to GMRes in the viewpoint of both convergence behavior and computational efficiency. Here we also take up the convergence for rigid-thin-object problems using HF.

2. Iterative Methods

2.1. *Krylov subspace methods and FMBEM*

For the solution of a linear system $\mathbf{Ax} = \mathbf{b}$, an iterative method gives successive approximations $\mathbf{x}_i$ at each step i from an initial approximation $\mathbf{x}_0$ in a systematic way, until the residual norm $\|\mathbf{r}_i\| = \|\mathbf{b} - \mathbf{Ax}_i\|$ sufficiently decreases. Krylov subspace methods require repeated calculation of matrix-vector products from the coefficient matrix $\mathbf{A}$ and vectors, which occupies quite large part of the total amount of operations. If $\mathbf{A}$ is a $N \times N$ dense matrix as generally obtained with the BEM, where N is DOF, the operation count for a matrix-vector product is $O(N^2)$. Thus, if the number of iteration is sufficiently smaller than N, iterative solvers remarkably reduce the total operation count for solving the system compared to direct solvers, which require $O(N^3)$ operations. In addition, the FMBEM computes matrix-vector products efficiently using the FMM, without computing the coefficient matrix $\mathbf{A}$ directly. Specifically, the coefficient matrix $\mathbf{A}$ is divided into two parts: a sparse matrix $\mathbf{A}_{\mathrm{near}}$, which is composed of the influence parts between only near elements, and a matrix $\mathbf{A}_{\mathrm{far}}$, composed of those beween far elements. Then a matrix-vector product $\mathbf{A}_{\mathrm{far}}\mathbf{u}$ $(= \mathbf{Au} - \mathbf{A}_{\mathrm{near}}\mathbf{u})$ is efficiently computed using the multipole expansion, where $\mathbf{u}$ is a vector. Since it is not necessary to keep the whole coefficient matrix $\mathbf{A}$, required memory is also reduced sharply. It is generally known that both the operation count and the required memory of the FMBEM are $O(N^a \log^b N)$, where $1 \leq a < 2$ and $b \geq 0$, and the values a and b depend on details of the programming and distribution of nodes (the boundary shape) of the problem.[27] In our implementation, the operation count of the iterative process, which occupies large amount of the total operation counts, and the required memory are both $O(N \log N)$, when the distribution of nodes can be regarded as two-dimensional such as a field in a room, and $O(N)$, when the distribution can be regarded as three-dimensional such as a field where many small scatterers are uniformly distributed. For more details of our implementation, see Ref. 5.

2.2. *Kinds of iterative solver*

Among the Krylov subspace iterative solvers applicable to the FMBEM, we use the following solvers.

- Lanczos type: CGS, BiCGStab, BiCGStab2 and GPBiCG, all of which are BiCG-like solvers.
- Arnoldi type: GMRes(l).

It is difficult to apply to the FMBEM some BiCG-like iterative solvers such as QMR and BiCG and the solvers for the normal equations such as CGNE and CGNR, because these solvers require matrices transpose.

2.3. *Stopping criterion for convergence*

The following equation is used as a stopping criterion for the linear system,

$$\frac{\|\mathbf{r}_i\|_2}{\|\mathbf{b}\|_2} = \frac{\|\mathbf{b} - \mathbf{A}\mathbf{x}_i\|_2}{\|\mathbf{b}\|_2} \le \varepsilon, \tag{1}$$

where $\mathbf{x}_i$ is the approximate solution vector of ith iteration, $\mathbf{r}_i$ is the residual vector of ith iteration, and $\|\cdot\|_2$ is the 2-norm. In this paper, $\varepsilon = 10^{-6}$.

2.4. *Number of iteration*

Solvers are different in operation count per iteration. Instead of the number of iteration, we use for the evaluation of convergence the number of matrix-vector products, the operation of which accounts for large part of the total operation.

2.5. *Improvement of convergence*

2.5.1. *Preconditioning*

Preconditioning is a well-known technique for improvement of convergence when one attempts to solve linear systems using iterative solvers. A matrix $\mathbf{M} = \mathbf{M}_1\mathbf{M}_2$, which approximates the coefficient matrix $\mathbf{A}$ in some way, transforms the original system into the following system with more favorable properties for iterative solution:

$$\mathbf{M}_1^{-1}\mathbf{A}\mathbf{M}_2^{-1}(\mathbf{M}_2\mathbf{x}) = \mathbf{M}_1^{-1}\mathbf{b}. \tag{2}$$

In general, a good preconditioner $\mathbf{M}$ should meet the following requirements: the preconditioned system should be easy to solve, and the preconditioner should be cheap to construct and apply. Various types of preconditioner have already been proposed, nevertheless there are not many types effective against non-Hermitian dense matrices, which are generally obtained with BEM. Moreover, the effective types are very few against calculation using FMBEM since the coefficient matrix $\mathbf{A}$ is not directly calculated in the FMBEM. For these reasons, we adopt only the following two preconditioning.

Diagonal preconditioning[11]

The preconditioner $\mathbf{M}$ consists of just the diagonal of the matrix $\mathbf{A}$. This preconditioning is applicable without extra memory and time, and easy to apply to the FMBEM. This preconditioning was reported to be effective against the thermal and elastic BEM with CGS, BiCGStab and GMRes,[36] whereas reported not so effective against the acoustic BEM.[13]

ILUT preconditioning (threshold based incomplete LU factorization)[37]

ILUT is a kind of preconditioning based on incomplete LU (ILU) factorization. ILUT employs a dual dropping strategy, where the threshold τ for dropping elements having small magnitude and the maximum number p of fill-in elements in each row of $\mathbf{L}$ and $\mathbf{U}$ are used to control the computational costs for factorization and application of the preconditioner. Preconditioning based on ILU factorization cannot be directly applied to FMBEM or other methods in which coefficient matrices are not calculated. Schneider *et al.* applied ILUT preconditioning to the sparse matrix $\mathbf{A}_{\mathrm{near}}$ (composed only of the influence parts between near elements), which is calculated in fast BEM such as FMBEM in the same manner of conventional BEM, and proved this way to apply ILUT to be effective against GMRes and CGNR.[38] We adopt the same way in this paper. For the parameters, we use a constant value $\tau = 10^{-5}$ and investigate only the effect of changing p, the value of which largely affects memory requirements. We will refer to this preconditioner as ILUT(p) in the rest of the paper.

2.5.2. *Initial shadow residual*

The initial shadow residual $\mathbf{r}_0^*$ is commonly required in BiCG-like solvers. $\mathbf{r}_0^*$ is an arbitrary vector, such that $\mathbf{r}_0^* \cdot \mathbf{r}_0 \neq 0$, e.g., $\mathbf{r}_0^* = \mathbf{r}_0$ (a usual setting). Fujino *et al.*[39] reported through numerical experiment great improvement of convergence by using some settings giving uniform random numbers to $\mathbf{r}_0^*$. We attempt one of these settings, "$\mathbf{r}_0 = \mathbf{b}$ and $\mathbf{r}_0^* =$ pseudorandom numbers" ($\mathbf{r}_0^* =$ random), as a method for improvement of convergence, because this was reported to be the most effective.[39]

3. Investigation of BiCG-Like Iterative Solvers

3.1. *Numerical setup*

For both interior and exterior problems, objects with simple and complex shapes are considered. Figure 1 shows four analysis models for the investigation. Theoretical solutions can be obtained for problems I-S (with rigid boundary) and E-S, which have simple shapes of boundary. All of these problems are employed from the benchmark problems in Ref. 40. The boundary conditions are as follows,[40] unless noted otherwise. Problems I-S and E-C: all surfaces are rigid. Problem I-C: an absorptive ceiling, and absorptive walls between the ceiling and real walls are assumed with the absorption coefficient $\alpha = 1$, because this hall has no ceiling. All the other surfaces are rigid. Problem E-S: all surfaces are vibrating with normal velocity, distribution of which is equal to that of particle velocity in the free field with a point source at the center of the cube. In all of these problems, the effect of changing the absorption coefficient α of boundaries is investigated. Only the real part of the acoustic impedance is assumed on absorption boundaries corresponding to the absorption coefficient α. Three formulations are used for the investigation: singular (SF) and hypersingular (HF) formulations, and formulation by Burton and Miller (BM) for exterior problems. Diagonal preconditioning and "$\mathbf{r}_0^* =$ random" are used as methods for improvement of convergence.

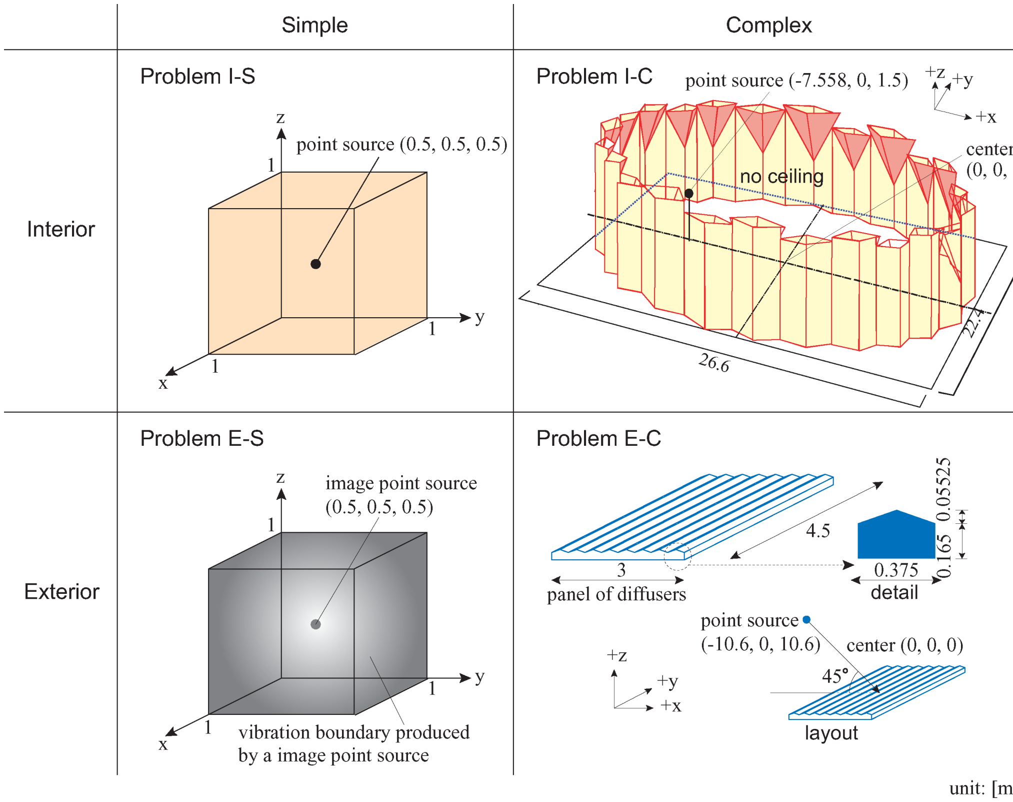


Fig. 1. Analysis models: a cube (I-S) and an auditorium (I-C) are interior problems, and a vibrating cube (E-S) and a diffuser (E-C) are exterior problems.

Boundaries are discretized using constant elements with width of less than 1/8 of the wavelength. The numerical items for calculation with the FMBEM based on the diagonal form are as follows: the truncation number N_{c} of terms in the infinite series for M2L translation coefficients is $N_{\mathrm{c}} = \lceil k(0.3D + 0.7r_{\mathrm{LM}}) + (r_{\mathrm{LM}}/D)\ln(kD+\pi)\rceil$, and the number K of quadrature points for integration over the unit sphere is $K = 2N_{\mathrm{c}}^2$, where k is the wave number, r_{LM} is the distance between centroids of cells (corresponding to multipole and local expansion points), and D is the diameter of a sphere circumscribing the cell whose centroid is M or L. These are determined through numerical experiments.[41] The initial values for iterative solvers are as follows, unless noted otherwise: the initial approximate solution of sound pressure vector $\mathbf{x}_0 = \mathbf{0}$, and $\mathbf{r}_0^* = \mathbf{r}_0$. The computation is executed with the supercomputer HITACHI SR8000. Parallel processing is not executed.

3.2. *Results and discussion*

3.2.1. *Accuracy*

Before comparing convergence behavior, we investigated the difference in accuracy between solvers, by using problems for which theoretical solutions can be obtained. Table 1 shows

Table 1. Differences in the results of calculations for the problem I-S, between using the mode summation method and using the FMBEM with unpreconditioned iterative solvers. $\varepsilon_{\rm ite}$ is defined as Eq. (3).

DOF	Freq. [Hz]	Formulation	$10\log_{10}\varepsilon_{\rm ite}$ [dB]			
			CGS	BiCGStab	BiCGStab2	GPBiCG
1,536	500	singular	−10.95	−10.95	−10.95	−10.95
		hypersingular	−15.64	−15.64	−15.64	−15.64
6,144	1,000	singular	−16.26	−16.26	−16.26	−16.26
		hypersingular	N.C.	−16.85	−16.85	−16.85
24,576	2,000	singular	N.C.	−8.24	−8.24	−8.24
		hypersingular	N.C.	−15.88	−15.88	−15.88
98,304	4,000	singular	−16.95	−16.95	−16.95	−16.95
		hypersingular	N.C.	−17.85	−17.85	−17.85

N.C.: not converging.

differences between theoretical solutions using the mode summation method (mode numbers up to 210 for each axis of rectangular coordinates) and iterative solutions using the FMBEM, in calculation of an interior problem I-S with rigid boundary. $\varepsilon_{\rm ite}$ in the table is defined as

$$\varepsilon_{\rm ite} = \frac{\sum_n \left| |p_{\rm th}(\mathbf{r}_n)|^2 - |p_{\rm ite}(\mathbf{r}_n)|^2 \right|}{\sum_n |p_{\rm th}(\mathbf{r}_n)|^2}, \tag{3}$$

where $p_{\rm th}(\mathbf{r}_n)$ and $p_{\rm ite}(\mathbf{r}_n)$ are sound pressures at the node position $\mathbf{r}_n$, obtained with the mode summation method and with the FMBEM, respectively. CGS does not converge in some cases, especially in HF, due to round-off errors. In the other cases, including CGS, no differences in accuracy between solvers are seen. Similar results were obtained in calculation of an exterior problem E-S, for which a theoretical solution also exists. From these results, it is considered that round-off errors do not raise the difference in calculation results of solvers when the solvers converge; all of the solvers give the same accuracy. In the paper below, therefore, we discuss the performance of iterative solvers only on the basis of the characteristics of the convergence.

3.2.2. *Convergence*

Effect of formulation

Figure 2 shows the number of matrix-vector products for an exterior problem E-S (simple shape) with unpreconditioned BiCGStab. Here $\varepsilon = 10^{-3}$ in the stopping criterion for convergence Eq. (1). The fictitious eigenfrequencies for SF and HF are also shown in the upper part of the figure. The convergence with HF and BM is slower than that with SF at all frequencies. The number of matrix-vector products for HF is greater at fictitious eigenfrequencies than at other frequencies, whereas relationship between the convergence with SF and its fictitious eigenfrequencies is not clearly observed. As for BM, no clear correspondence

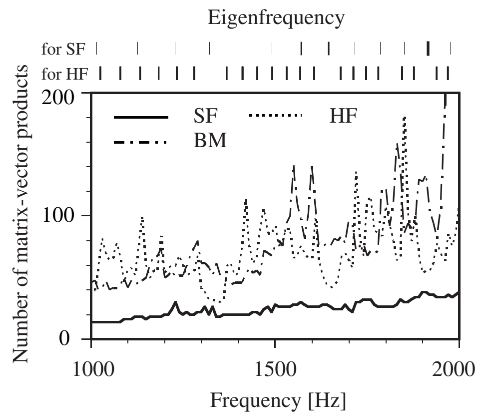


Fig. 2. Effect of different kinds of formulation on the number of matrix-vector products for the problem E-S, using FMBEM in SF, HF, and BM with unpreconditioned BiCGStab. DOF is 6,144.

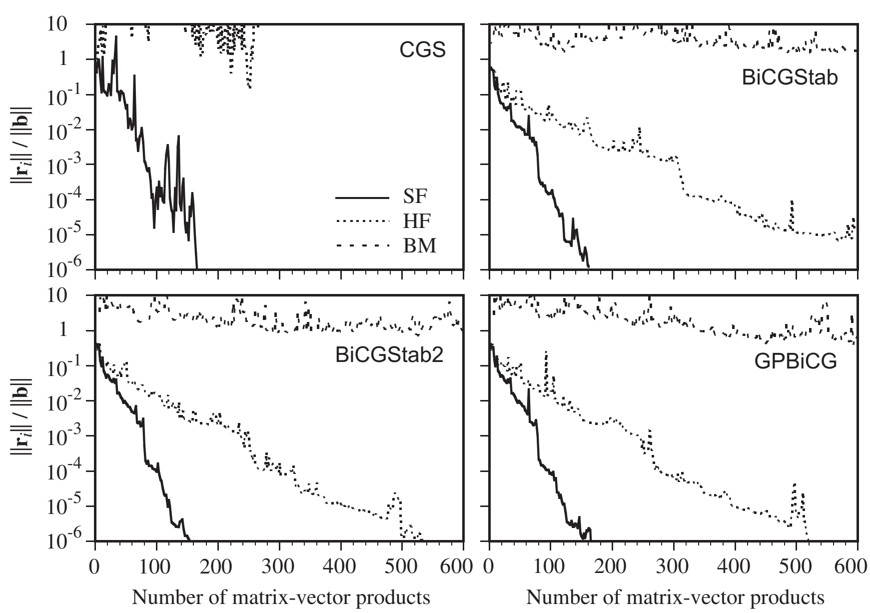


Fig. 3. Effect of different kinds of iterative solver on the iteration residual for the problem E-S at 4 kHz, using FMBEM in SF, HF, and BM. DOF is 98,304.

between the increase of the number of matrix-vector products and fictitious eigenfrequencies is seen, similarly to the case of SF. Figure 3 shows the history of the residual norms for the same problem with different unpreconditioned iterative solvers, using three kinds of formulations. It is also seen that convergence of the solver with HF and BM is clearly slower than that with SF, whatever solver is used. The convergence with BM is the slowest. The same results were seen in calculations of the other problems, and also reported in Ref. 13. From these results, it is confirmed that matrices obtained with HF are generally ill-conditioned compared to those with SF, independently of the kind of BiCG-like solver, and that BM matrices inherit the ill condition from HF matrices. In SF, no difference between solvers can be seen, except for oscillations of CGS. There was the same tendency in the calculation of the other problems.

Effect of iterative solvers

It is seen in Fig. 3 that the convergence behavior of CGS is irregular, which leads to divergence in HF and BM. BiCGStab, BiCGStab2 and GPBiCG have similar convergence behavior, whereas BiCGStab has a more slower convergence than GPBiCG and BiCGStab2 in HF. Figure 4 shows the history of the residual norms for the exterior problem E-C (complex shape). In contrast with the simple-shape problem E-S, CGS is the fastest to converge among the four solvers in BM. Generally, an ith degree residual of a BiCG-like solver can be expressed as $\mathbf{r}_i = P_i(\mathbf{A})\mathbf{r}_0$, where P_i is the residual polynomial. An ith degree residual polynomial in CGS ($P_{i,\mathrm{CGS}}$) is the square of that in BiCG ($P_{i,\mathrm{BiCG}}$), i.e. $\mathbf{r}_i^{\mathrm{CGS}} = P_{i,\mathrm{CGS}}(\mathbf{A})\mathbf{r}_0 = P_{i,\mathrm{BiCG}}^2(\mathbf{A})\mathbf{r}_0$, and this square property of $P_{i,\mathrm{CGS}}$ can contribute to either acceleration of convergence or propagation of divergence.[42] These facts show that CGS is an unstable solver depending on problems, with a possibility of fast convergence for some cases such as exterior problems solved with Burton-Miller formulation. It is also seen in Fig. 4 that BiCGStab has a slower convergence than GPBiCG and BiCGStab2 in BM.

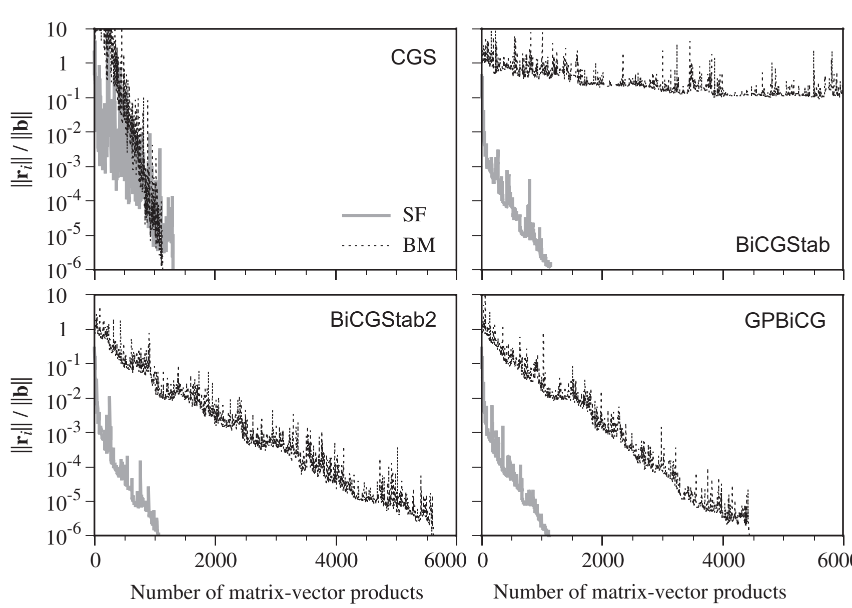


Fig. 4. Effect of different kinds of iterative solver on the iteration residual for the problem E-C at 1 kHz, using FMBEM in SF and BM. DOF is 24,616.

Effect of boundary shapes

Figure 5 shows the history of the residual norms for interior problems I-S (simple shape) and I-C (complex shape), with different boundary conditions. DOFs are nearly the same between two problems. Slower convergence is seen for the problem with a complex shape (I-C) than with a simple shape (I-S), when both of the problems are under the same boundary condition. The same results were also seen with the other solvers or in the case of exterior problems, and also reported in Ref. 13.

Effect of boundary conditions

It is seen in Fig. 5 that the larger the absorption coefficient α is, the faster the convergence is, not depending calculated problems and used solvers. Figure 6 shows the effect of the small change of the absorption coefficient around $\alpha = 0$. It is seen that only a little absorption coefficient is sufficiently effective in the improvement of convergence in comparison with the case of rigid boundary ($\alpha = 0$).

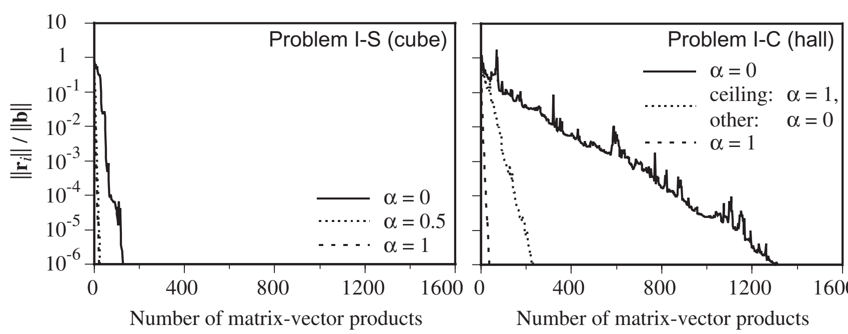


Fig. 5. Effect of shapes and boundary conditions on the iteration residual, using FMBEM in SF with unpreconditioned GPBiCG. DOF is 24,576 (cube) and 24,514 (hall). α is the absorption coefficient on the surfaces.

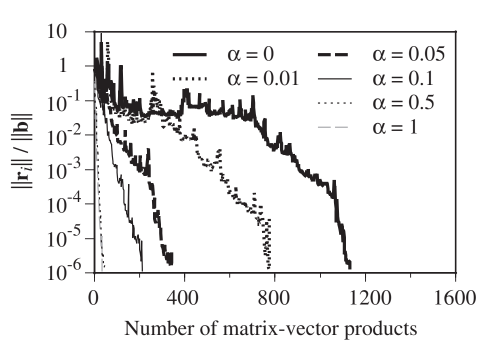


Fig. 6. Effect of absorption coefficients α on the iteration residual for the problem I-C at 63 Hz, using FMBEM in SF with unpreconditioned BiCGStab. DOF is 6,110.

Effect of methods for improvement of convergence

Figures 7 and 8 show the history of the residual norms for the exterior problem E-S using three formulations, and the effect of the setting where pseudorandom numbers are given to the initial shadow residual ("$\mathbf{r}_0^* =$ random"). In HF and BM, the convergence behavior is quite worse with a usual setting ("$\mathbf{r}_0^* = \mathbf{r}_0$"), independent of solvers, and divergence is seen in the case of CGS. In contrast, remarkable improvement of convergence is seen with the setting "$\mathbf{r}_0^* =$ random" only in the case of CGS, which results in much better convergence than the other solvers in BM. Therefore, CGS with the setting "$\mathbf{r}_0^* =$ random" is a good choice for calculation of exterior problems with Burton-Miller formulation. The reason why this setting is effective only against CGS is considered as follows: a residual polynomial in CGS is the square of that in BiCG, as mentioned above, whereas a residual polynomial in the other solvers have a term for stable convergence, which prevents acceleration of convergence.

As for diagonal preconditioning, the effect on convergence is hardly seen in many cases, as reported also in Ref. 13. However, there are some exceptions: the cases of exterior problems discretized with various-sized elements in HF or BM. Figure 9 shows examples of these

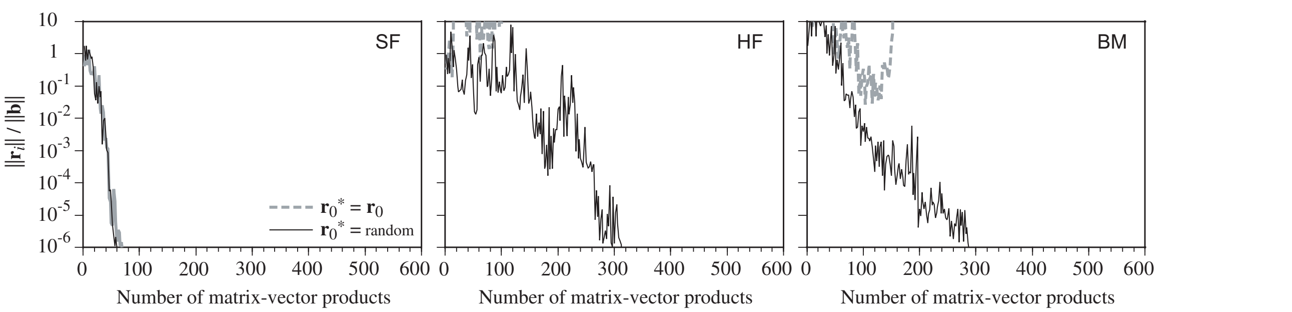


Fig. 7. Effect of initial shadow residual $\mathbf{r}_0^*$ on the iteration residual for the problem E-S at 2 kHz, using FMBEM with unpreconditioned CGS. DOF is 24,576.

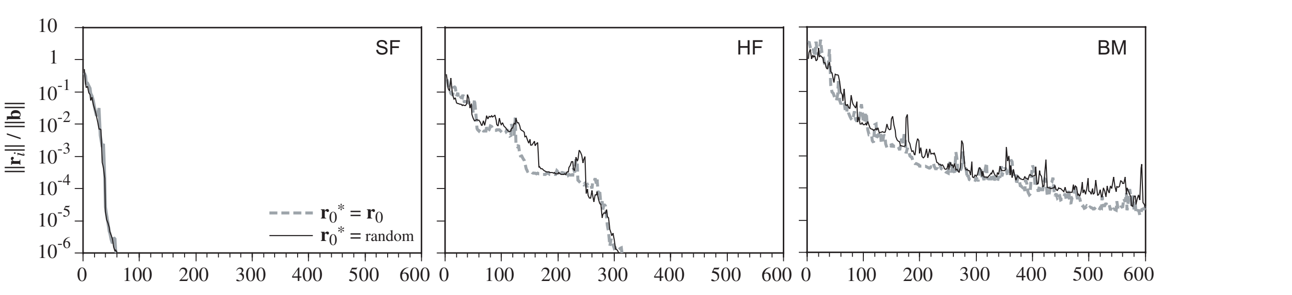


Fig. 8. Effect of initial shadow residual $\mathbf{r}_0^*$ on the iteration residual for the problem E-S at 2 kHz, using FMBEM with unpreconditioned GPBiCG. DOF is 24,576.

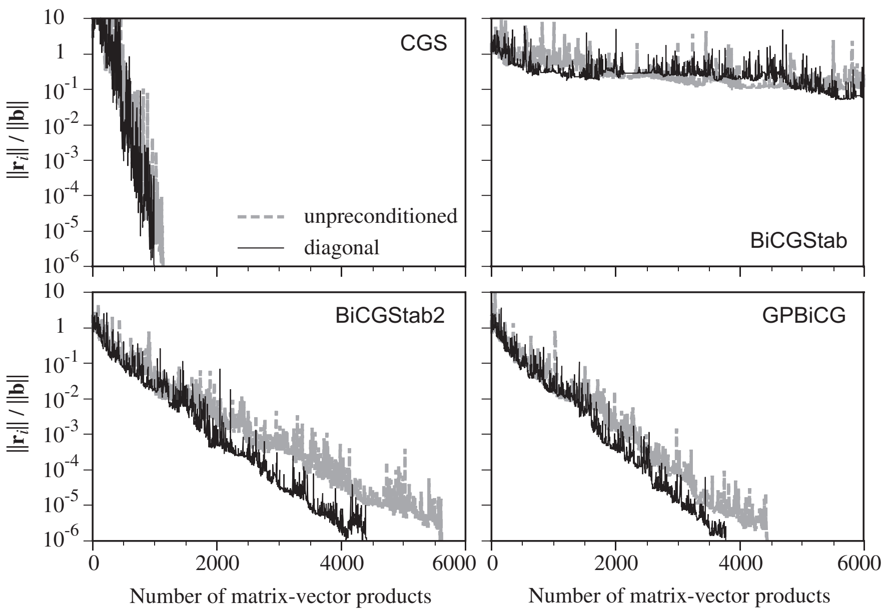


Fig. 9. Effect of diagonal preconditioning on the iteration residual for the problem E-C at 1 kHz, using FMBEM in BM. DOF is 24,616.

cases. One of the reasons for the improvement of convergence in these cases is considered as follows: the values of the diagonal of the matrix are not the same because of the different size of elements, and these values easily differ from each other due to the shape or area of each element, because the kernel in the integral of elements is hypersingular.

Summary of convergence behavior

Table 2 shows a brief summary of the results for convergence behavior of BiCG-like solvers in this section.

Table 2. Summary of convergence behavior with BiCG-like iterative solvers. $\mathbf{p}_0^*$ is the initial shadow residual. A $<$ B means that the number of matrix-vector products with A is smaller than that with B.

	Interior problem	Exterior problem
Formulation	SF $<$ HF	SF $<$ HF $<$ BM
Solver	GPBiCG $\leq$ BiCGStab2 $<$ BiCGStab $<$ CGS†	
Shape	simple $<$ complex	
Boundary condition	absorptive $<$ rigid	
Diagonal preconditioning	not effective with SF	a little effective with HF and BM
$\mathbf{p}_0^*$ = random	quite effective only with CGS	

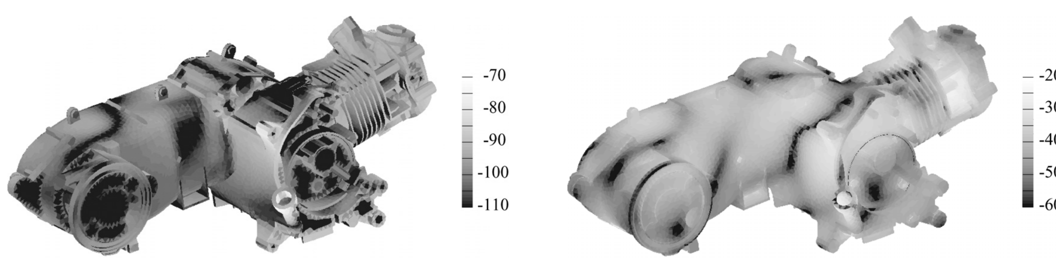
†CGS is the fastest for some exterior problems with BM, especially complex-shape problems.

4. Comparison Between BiCG-Like Solvers and GMRes

In the previous section, GPBiCG and BiCGStab2 were stable, and CGS was the fastest for Burton-Miller formulation. Here we compare the convergence of GPBiCG and CGS with that of GMRes, which is known as a robust solver. We also investigate the effect of ILUT preconditioning — reported to be effective against GMRes in Ref. 13 — on GPBiCG and CGS. Moreover, we investigate the convergence of these solvers for the calculation of rigid-thin-object problems (with 0 thickness) using HF.

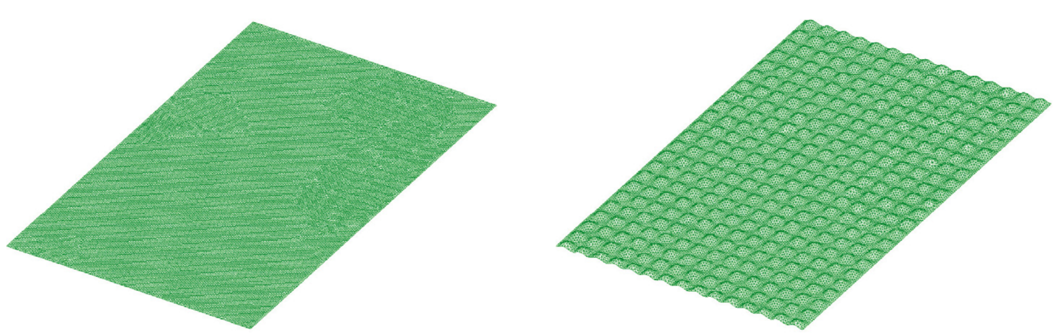
4.1. *Numerical setup*

We again adopt the small hall (problem I-C) and the diffuser (problem E-C) shown in Fig. 1 as the interior and exterior problems, respectively, and a new engine model having a much complex shape as the other exterior one (problem E-C2). All surfaces of the engine model are assumed vibration boundaries, given normal vibrating velocity obtained with structural analysis. Figure 10(a) shows the shape of the engine model and vibration velocity distribution used as a boundary condition. In addition, we adopt two kinds of 0-thick rectangular rigid panel shown in Fig. 11 as the thin-object problems: (a) a plane panel (problem Et-S),



(a) (b)

Fig. 10. An engine model (E-C2): (a) relative vibration velocity level distribution, and (b) relative SPL distribution at 1,977 Hz, calculated using FMBEM based on Burton-Miller formulation (BM). DOF is 42,152.



(a) (b)

Fig. 11. Analysis models: (a) a plane rigid panel (Et-S), and (b) a 2D sinusoidal-shaped rigid panel (Et-C). The size of the panels and the condition of incidence from a point source is the same as the problem E-C. DOF is 45,620 for both models.

and (b) a 2D sinusoidal-shaped panel (problem Et-C). We adopt for the interior problem the singular formulation (SF), with which the calculation converges stably and rapidly in the previous section, Burton-Miller formulation (BM) for the exterior problem, and hyper-singular formulation (HF) for the rigid-thin-object problems. ILUT(p) preconditioning is used as a method for improvement of convergence. We investigated the effect of the setting "$\mathbf{r}_0^* =$ random" on the convergence of CGS, only on which this setting was effective in the previous section, and observed no improvement of convergence in all cases. Thus, the effect of the setting "$\mathbf{r}_0^* =$ random" is not shown in this section. Conditions of calculation are the same as the previous section. Parallel processing is not executed. The computation is executed with VT64 Opteron Workstation (Opteron 2 GHz).

4.2. *Results and discussion*

4.2.1. *Analysis results*

Figure 10(b) shows sound pressure level distribution for the engine model, calculated in BM with the vibration velocity distribution shown in Fig. 10(a). The difference in sound pressure level due to the kinds of solver and of preconditioning is less than 0.01 dB.

4.2.2. *Convergence*

Interior problems

Figure 12 shows the history of the residual norms for the small hall (problem I-C) calculated in SF. Unpreconditioned GPBiCG converges faster than unpreconditioned GMRes(no restart) (GMRes(∞)), and they show nearly the same convergence behavior when these solvers are preconditioned by ILUT. These tendencies are almost the same when all surfaces are assumed to be rigid (more difficult to converge), in both preconditioned and unpreconditioned cases. Taking into account these results and the difficulty of handling arising from restart in GMRes, one can say that GPBiCG is a good solver for calculation of interior problems. To investigate the convergence behavior for problems having more complex shape

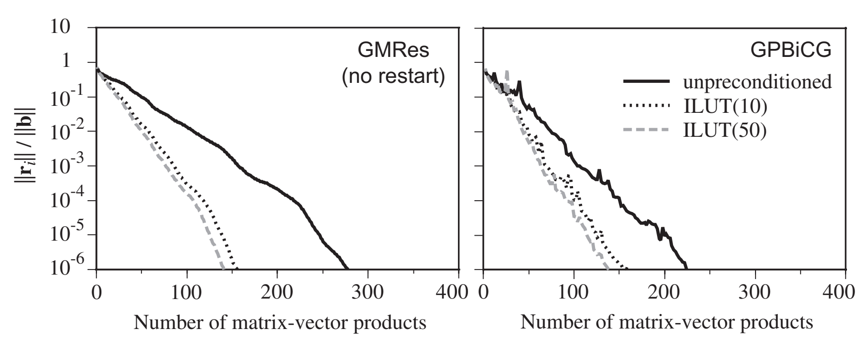


Fig. 12. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem I-C at 125 Hz, using FMBEM in SF. DOF is 24,514. In this case, improvement of convergence was not seen when p was greater than about 50.

boundaries, here we assume the engine model (E-C2) mentioned above as an interior problem and use it for investigation. All surfaces are assumed vibration boundaries, which is the same as the analysis as an exterior problem. This interior problem has a much complex shape and no absorptive boundaries; the results in the previous section suggest that this is a special bad case for convergence among all interior problems. The results are shown in Fig. 13. In contrast to the problem I-C, GMRes(∞) is much faster than GPBiCG, independently of the value p. The effect of ILUT(p) increases with p for both solvers. From these results, GMRes(∞) with ILUT(p) preconditioning is generally recommended for calculation of interior problems. However, taking into account that this problem is the special bad one for convergence, one can state that GPBiCG with ILUT(p) is sufficiently good for many practical problems.

Exterior problems

Figure 14 shows the history of the residual norms for the diffuser (problem E-C) calculated in BM. Without preconditioning, CGS is the fastest among all solvers including GMRes(∞).

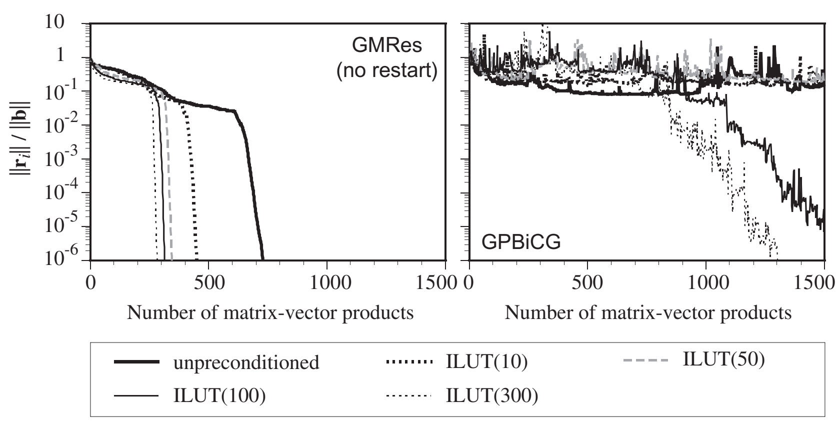


Fig. 13. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem E-C2 (as an interior problem) at 3 kHz, using FMBEM in SF. DOF is 42,152.

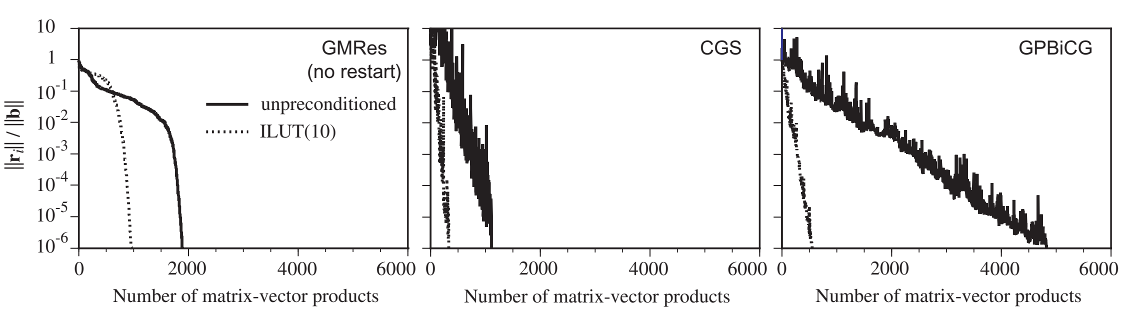


Fig. 14. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem E-C at 1 kHz, using FMBEM in BM. DOF is 24,600. In this case, improvement of convergence was not seen when p was greater than about 10.

With ILUT preconditioning, CGS is also the fastest, and GMRes(∞) is the slowest. This confirms the tendency in the previous section that CGS is faster to converge than other solvers for calculation of exterior problems in Burton-Miller formulation.

Figure 15 shows the history of the residual norms for the engine (problem E-C2) calculated in BM. Only in this case, all unpreconditioned solvers do not converge at all, which again shows the difficulty of convergence in BM. In contrast, ILUT(p) remarkably improves the convergence of all solvers. The effect of ILUT(p) increases with p. As for GMRes(∞), even small p greatly improves the convergence, making GMRes(∞) the fastest among the solvers. Between BiCG-like solvers, CGS is faster than GPBiCG. From the viewpoint of convergence behavior only, GMRes(∞) with ILUT(p) preconditioning is generally recommended for calculation of exterior problems in Burton-Miller formulation, while CGS with ILUT(p) may be better than GMRes(∞) for problems which do not have much complex shapes.

Figure 16 shows the effect of restart number l for GMRes(l) on the iteration residual. The residual norms stagnate from every restart point.[38] Without restart (Fig. 15), p does

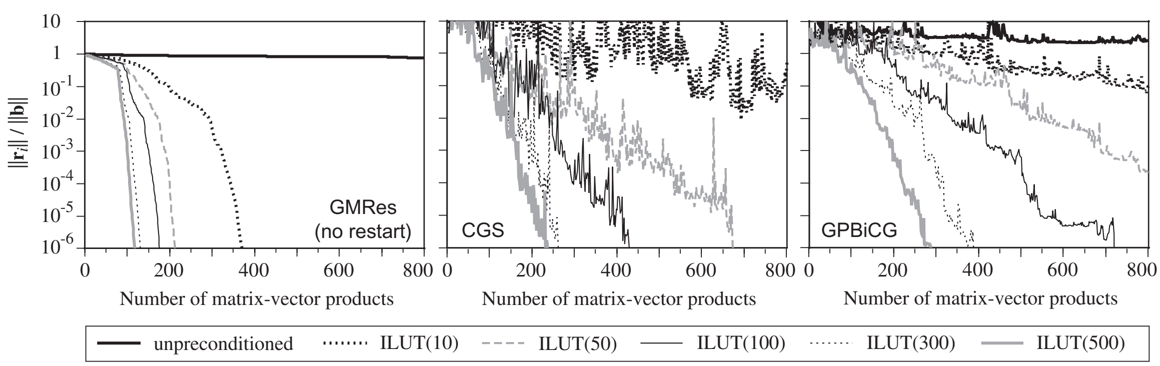


Fig. 15. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem E-C2 at 3 kHz, using FMBEM in BM. DOF is 42,152.

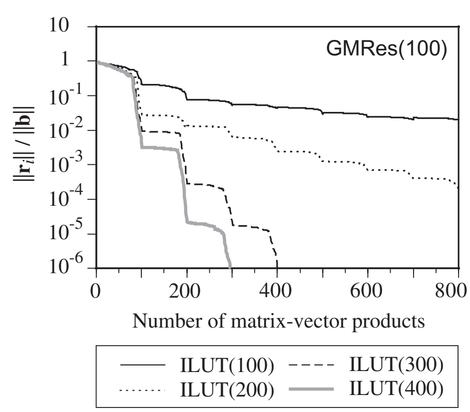


Fig. 16. Effect of restart number l for GMRes(l) on the iteration residual for the problem E-C2 at 3 kHz, using FMBEM in BM. $l = 100$, and DOF is 42,152.

not affect the convergence behavior so much when $p \geq 100$, whereas with restart (Fig. 16), the convergence behavior from each restart point is quite different; the smaller p is, the slower the convergence is. Consequently, this results in slower convergence of GMRes(l) than CGS (Fig. 15) in some cases. These results indicate that avoiding restart is important in GMRes(l) from the viewpoint of reducing computational time, and that appropriately-preconditioned BiCG-like solvers such as CGS preconditioned by ILUT possibly converge faster than GMRes(l) even for complex-shape problems, if restart cannot be avoided in GMRes(l) due to the restriction of computer memory.

Rigid-thin-object problems

Figures 17 and 18 show the history of the residual norms calculated in HF, for the plane panel (problem Et-S) and for the sinusoidal-shaped panel (problem Et-C), respectively. The convergence for both problems is remarkably faster than the exterior problems with nearly the same DOF (Fig. 15) and with less DOF (Fig. 4). The big difference between these exterior problems and the thin-object problems is the existence of inner space, and calculation of thin-object problems in HF has nothing to do with the fictitious eigenfrequency difficulties.

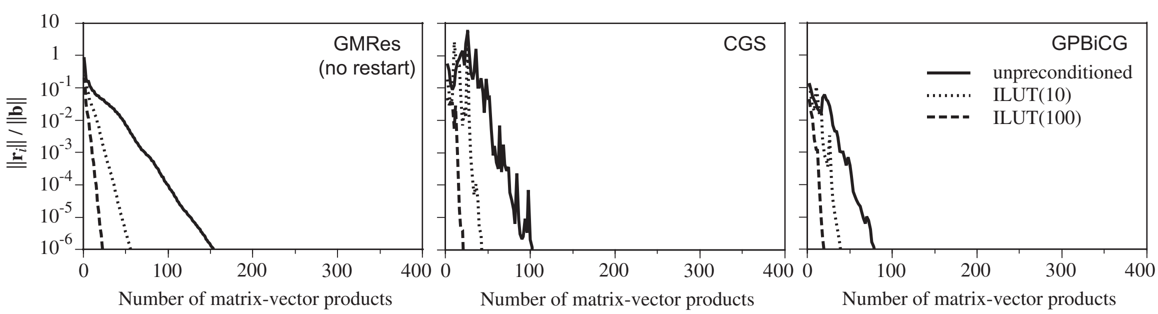


Fig. 17. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem Et-S at 2 kHz, using FMBEM in HF. DOF is 45,620. In this case, improvement of convergence was not seen when p was greater than about 100.

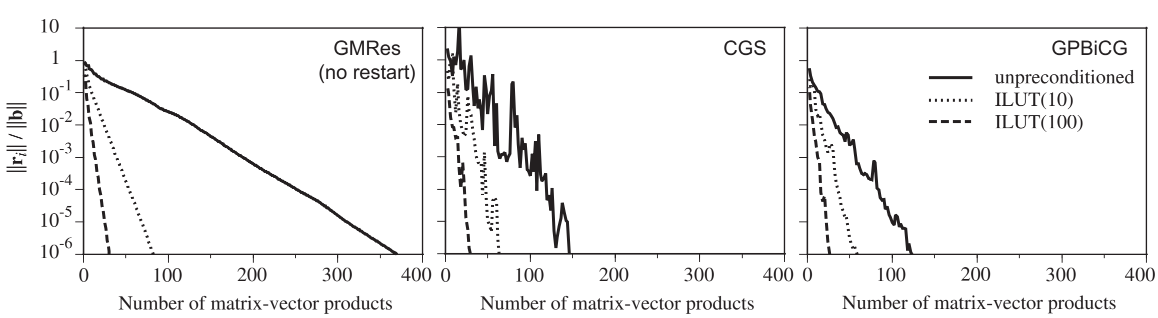


Fig. 18. Effect of the number of fill-in elements p for ILUT(p) on the iteration residual for the problem Et-C at 2 kHz, using FMBEM in HF. DOF is 45,620. In this case, improvement of convergence was not seen when p was greater than about 100.

This probably relates to the faster convergence for thin-object problems. When unpreconditioned, CGS and GPBiCG show faster convergence than GMRes(∞), and even when preconditioned by ILUT(p), these BiCG-like solvers show convergence as fast as or faster than GMRes(∞). These results indicate that the linear equation obtained in calculation of rigid-thin-object problems with HF is relatively easy to converge, and that BiCG-like solvers are better than GMRes from the viewpoint of convergence behavior and ease to handle (free from restart).

Summary of convergence behavior

Table 3 shows a brief summary of the results for convergence behavior in this section.

4.2.3. *Computational efficiency*

Figure 19 shows details of computational time for the problem E-C2 (engine) calculated in BM. While on the one hand the computational time for iterative process decreases with increase in p of ILUT(p) for all solvers, the computational time for ILUT preconditioning increases with p; thus it is difficult to determine the optimum p minimizing the total computational time before calculation. Additional required memory for ILUT preconditioning in this calculation was 170 MB when $p = 100$ (6% of the total memory), and 844 MB when $p = 500$ (25% of the total memory); the amount cannot be bypassed when p is a large value.

Table 3. Summary of convergence behavior with GMRes(∞), CGS and GPBiCG, with/without ILUT preconditioning. A < B means that the number of matrix-vector products with A is smaller than that with B.

Shape	ILUT	Interior problem	Exterior problem	Rigid thin problem
Simple	without	GPBiCG $<$ GMRes	CGS $<$ GMRes $<$ GPBiCG	GPBiCG $\leq$ CGS $<$ GMRes
Simple	with	GPBiCG $\approx$ GMRes	CGS $<$ GPBiCG $<$ GMRes	GPBiCG $\approx$ CGS $\approx$ GMRes
Complex	without	GMRes $\ll$ GPBiCG†	GMRes $\ll$ GPBiCG $<$ CGS	GPBiCG $\leq$ CGS $<$ GMRes
Complex	with	GMRes $<$ GPBiCG†	GMRes $<$ CGS $<$ GPBiCG	GPBiCG $\approx$ CGS $\approx$ GMRes

†These are results for a special problem, not realistic as an interior one, which has a very complex shape without absorption.

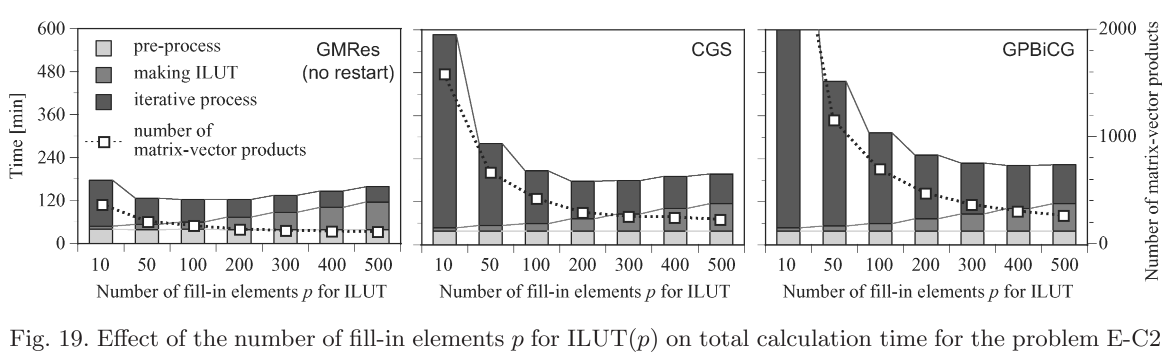


Fig. 19. Effect of the number of fill-in elements p for ILUT(p) on total calculation time for the problem E-C2 at 3 kHz, using FMBEM in BM. DOF is 42,152.

The increasing memory with the number of iteration in GMRes(l) was 136 MB (5% of the total memory) when $l = 200$, with which calculation can be done without restart if $p \geq 100$ of ILUT(p) was applied; meanwhile the memory reached 3.8 GB (60% of the total memory), if $l = 5,000$ (about 1/10 of the DOF 42,152) was adopted as a safe setting for avoiding restart. This increase of memory requirements is critical in many practical cases, especially when one can use only laptop computers or those with similar performance. It is difficult to determine an appropriate l for a good balance between computational time and required memory before calculation; from this point, though being inferior to the optimized GMRes, CGS preconditioned by ILUT is also a good choice when calculating exterior problems in Burton-Miller formulation.

5. Conclusions

Convergence behavior of Krylov subspace iterative methods towards the linear systems with the 3D fast multipole BEM, and the effect of methods for improvement of convergence was investigated through numerical experiments. Conclusions are summarized as follows:

(1) In BE analyses, there is no effective solver against all acoustic problems in convergence. One should select a solver according to the problem type.
(2) It has been confirmed that the convergence in hypersingular and Burton-Miller formulation is slower than that in basic singular formulation and that the greater the absorption coefficients of boundaries are, the faster the convergence is, independently of the type of problem and solver. It has been also confirmed that calculation of a problem having a complex shape converges slower than a problem having a simple shape, when DOF is nearly the same and the same iterative solver is used.
(3) GPBiCG and BiCGStab2 showed more stable convergence behavior than others among BiCG-like solvers. These solvers are useful for solving interior problems in singular formulation.
(4) Although CGS is an unstable solver depending on problems, it converges much faster than other BiCG-like solvers for some cases, especially exterior problems solved with Burton-Miller formulation.
(5) The setting "the initial shadow residual equals pseudorandom numbers" was effective only against CGS in improvement of convergence in the present investigation.
(6) Diagonal preconditioning had little effect on improvement of convergence in many cases, whereas some effect was observed in the cases of exterior problems discretized with various-sized elements calculated in hypersingular or Burton-Miller formulation.
(7) ILUT preconditioning improved the convergence for problems having complex shapes such as an engine model, independently of the solver. When solving exterior problems with greatly complex shapes in Burton-Miller formulation, all solvers hardly converged without preconditioning, whereas the convergence behavior was remarkably improved with ILUT preconditioning. In these cases GMRes(no restart) was the fastest, whereas CGS is one of the good choices when restart is required for GMRes due to the restriction of computer memory.

(8) The convergence for rigid-thin-object problems calculated in hypersingular formulation was faster than exterior problems with nearly the same DOF. BiCG-like solvers such as GPBiCG and CGS showed faster convergence than GMRes, and are recommended as good solvers for these problems, from the viewpoint of ease of handling compared to GMRes.

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