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Numerical implementation of the EDEM for modified Helmholtz BVPs on annular domains

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ABSTRACT

In a recent paper by the current authors a new methodology called the Extended-Domain-Eigenfunction-Method (EDEM) was proposed for solving elliptic boundary value problems on annular-like domains. In this paper we present and investigate one possible numerical algorithm to implement the EDEM. This algorithm is used to solve modified Helmholtz BVPs on annular-like domains. Two examples of annular-like domains are studied. The results and performance are compared with those of the well-known boundary element method (BEM). The high accuracy of the EDEM solutions and the superior efficiency of the EDEM over the BEM, make EDEM an excellent alternate candidate to use in the animation industry, where speed is a predominant requirement, and by the scientific community where accuracy is the paramount objective.

1. Introduction

In a recent paper [1] the authors proposed a semi-analytic method for solving elliptic boundary value problems (EBVPs) on annular-like domains with boundaries involving complex geometry, the Extended-Domain-Eigenfunction method or EDEM. The method is based on the concept of considering the domain Ω of the original problem to be part of a much larger domain that possesses greater symmetry. In this paper, we investigate one possible numerical implementation of EDEM.

In the original EDEM paper, the method was also posed as an alternative to more traditional methods such as the Finite Element Method (FEM) [2,3], the Boundary Element Method (BEM) [3–11], the Finite Difference Method (FDM) [12] and the Boundary Point Method (BPM) [13–17]. These types of numerical methods are required to solve EBVPs when an analytic or semi-analytic solution cannot be obtained, due to the complexity of the boundary. EDEM provides an alternative which is much easier to implement, less numerically intense and more accurate, and due to its semi-analytic nature it also provides greater insight into the actual problem.

The methodology of EDEM [1] involves formulating a related problem on an appropriate larger or extended domain. Using the inherent symmetry of the new domain, an eigenfunction solution is generated for the related problem on the extended domain using standard techniques such as separation of variables. The problem can then be solved using a

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variety of numerical techniques (e.g. a collocation method). Finally, this solution is restricted to the original domain to obtain the solution of the original problem. In the original paper [1], we provide a formal statement of EDEM and detailed discussion about the criteria and boundary conditions that allow for the method to be used for the case of the Laplace operator.

Recently, Shankar [18,19] independently proposed a similar method to EDEM, based on the method of eigenfunction expansions (MEE). Much like EDEM, a BVP on a complicated geometric domain is embedded in a larger domain endowed with a regular boundary and a complete set of eigenfunctions, allowing for an eigenfunction solution to be formulated. The solution of this embedded problem could then be used to effect the solution of the original problem. The method was applied to solve a series of simply and multiply connected domains for a variety of problems. However, as Shankar [18] admitted in his paper some important theoretical issues were not addressed and also the method assumed that the original domain was convex and that a nice extension for the solution existed. Some of these theoretical issues have now been addressed in [1]. Shankar also left open questions regarding the comparison of the method with other solution techniques, in particular the Boundary Element Method. Issues about computational efficiency and ease of implementation were only briefly discussed. This latter issue is what we address in the present publication. As was pointed out in [1], EDEM is not limited to the particular numerical implementation advocated in this paper, i.e., a collocation based approach. Shankar has demonstrated this fact by using a least squares approach. This difference in numerical schemes demonstrates a very important fact that there is no unique numerical approach associated with the theoretical method.

When numerical implementations are considered, they can be seen to overlap with another semi-analytic approach known as the Trefftz method [20–22]. The term Trefftz method is used to describe a series of diverse and different methods, all derived from Trefftz's original idea [20]. These methods also utilise eigenfunctions of the differential operator to construct a finite sum approximation to the EBVP. Therefore, overlap between our implementation and earlier ones occurs when the eigenfunction solution is truncated in the EDEM algorithm. In the literature, the Trefftz methods are referred to as collocation based Trefftz methods. An in-depth discussion of these methods can be found in [23]. It is important to point out that the EDEM methodology as outlined in [1] provides a theoretical basis even for these Trefftz methods and other related numerical schemes for problems on annular-like domains. In light of this overlap and to avoid any ambiguity in the literature we shall refer to the implementation followed in this paper as the EDEM–Trefftz algorithm.

A significant volume of literature and research has been published on the Trefftz method. In particular over the past decade there has been a resurgence in interest in the method. The potential for a mesh free algorithm with fast computational times makes it an area of interest both within and outside of the academic community. However, traditionally the Trefftz method is seen to be limited by the scope of domains it can handle. Typically, it has been restricted to singly connected domains of simple geometry, whereas as FEM and the BEM have been accepted as being more applicable to generic domains. Huang and Shaw [24] improved on the standard collocation Trefftz method by applying the embedding integral approach to the eigenfunction expansion method. By doing so and choosing partitions based on the problem's domain, they were able to solve Laplace and Helmholtz problems on a variety of different domains. Also, some recent works have expanded on this idea by looking into multi-pole Trefftz methods for solving Helmholtz and modified Helmholtz problems on multiply connected circular domains [25,26].

In this paper we will investigate and implement an EDEM–Trefftz numerical algorithm. We consider and solve EBVPs for the case of the modified Helmholtz equation

$$\Delta u - \kappa^2 u = 0, \quad x \in \Omega. \tag{1}$$

This elliptic operator is chosen for two reasons. Firstly, the modified Helmholtz operator is seldom studied in this context, while most of the literature dedicated to semi-analytic numerical solutions of EBVPs has concentrated on other more well-known classical PDE operators such as the Laplace [21,27–32], Poisson [21,33,34] and standard Helmholtz [21,35–37,26,25] equation, even in the case of the Trefftz method. This study will therefore add to the literature on this modified Helmholtz operator [22,38,10,25] as well as provide insight into the recently proposed EDEM and much older Trefftz method. This will complement the current body of work looking at applications of the BEM to the modified Helmholtz BVP [8–11]. Second, the extendability criteria (including the characterisation of the operator which maps the original boundary conditions for the original problem to the new boundary conditions) have been fully explored for the case of the Laplace operator. Since the modified Helmholtz equation is closely related to the Laplace equation (it degenerates to the Laplace equation as $\kappa \to 0$), the extendability criteria and the arguments to determine them are very similar (although we do not include them here). As there are no known closed form solutions for the domains we considered, we compare the EDEM–Trefftz algorithm with the BEM since these two techniques share some similarities.

The organisation of this paper is as follows. In Section 2 we review and present an overview of the EDEM methodology for the case of a modified Helmholtz EBVP on an annular domain. We present a general example for a 2D annular domain, formulate the related problem on the extended domain and present its general solution. In Section 3, we outline the EDEM–Trefftz algorithm. Finally in Section 4, we present numerical results for two examples of annular domains involving elliptic and square inner boundaries. These results are compared for both numerical accuracy and simulation efficiency with those of the BEM using linear elements. It is shown that the EDEM–Trefftz algorithm not only produces good results which appear to be more accurate than those of the BEM, it also demonstrates a significant advance on the BEM runtimes. We also show that similar findings are found to be consistent for boundary conditions that violate the EDEM domain extension conditions [1].

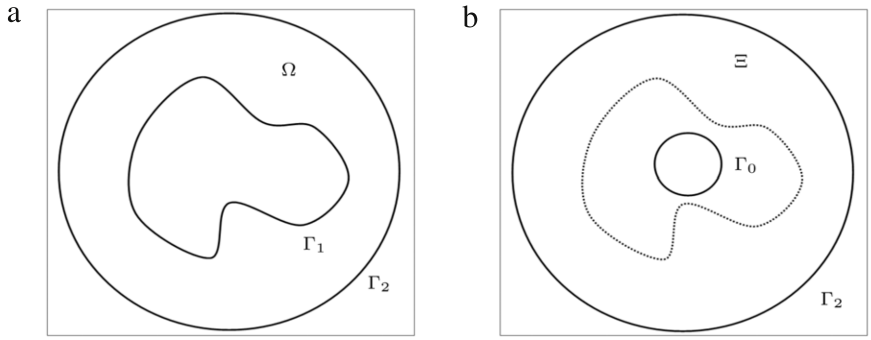

Fig. 1. Schematic illustrations of (a) original domain Ω and (b) extended domains Ξ to which the EDEM is applied: region between a 2D, closed smooth surface enclosed by a 2D circle.

2. Extended-Domain-Eigenfunction-Method EDEM

In this section we present briefly EDEM for the case of the modified Helmholtz operator based on the methodology presented in [1].

Consider the following EBVP involving the modified Helmholtz operator defined in the region $\Omega \subset \mathbb{R}^2$

$$\text{problem A} \begin{cases} \Delta v - \kappa^2 v = 0 & x \in \Omega, \\ v\,|_{\Gamma_1} = \eta_1, \\ v\,|_{\Gamma_2} = 0, \end{cases} \tag{2}$$

where $\eta_1 \in L^2(\Gamma_1)$ and Ω is the problem's annular domain and is given as the region enclosed by the boundaries $\Gamma_1 : r = t_1(\theta)$ and $\Gamma_2 : r = t_2(\theta) = b$; see Fig. 1(a). Both the outer circular boundary Γ_2 and the inner boundary Γ_1 are centered at $(0, 0)$. We assume that the inner boundary and the condition applied there are such that an extension to a larger domain is allowed (see [1] for a description of the extendability criteria). The first step of EDEM is to consider the original domain Ω to be enclosed in a larger domain which we denote as the extended domain $\Xi \supset \Omega$ (Fig. 1(b)). The extended domain is bounded by two concentric circles $\Gamma_0 : r = t_0(\theta) = a \leq \min(t_1(\theta))$ and Γ_2. This new extended domain contains much greater symmetry. We then formulate a related BVP

$$\text{problem B} \begin{cases} \Delta u - \kappa^2 u = 0 & x \in \Xi, \\ u\,|_{\Gamma_0} = \eta_0, \\ u\,|_{\Gamma_2} = 0, \end{cases} \tag{3}$$

in which η_0 is initially unknown.

Due to the symmetry of the extended domain, we can obtain an eigenfunction solution to problem B, using the separation of variables technique. The most general solution to the modified Helmholtz equation satisfying homogeneous Dirichlet conditions on Γ_2 is

$$u(r, \theta) = \sum_{m=0}^{\infty} \left(K_m(\kappa r) - \frac{K_m(\kappa b) I_m(\kappa r)}{I_m(\kappa b)} \right) (A_m \cos(m\theta) + B_m \sin(m\theta)), \tag{4}$$

where I_m and K_m are modified Bessel functions of the first and the second kind, respectively. Normally, the above equation is interpreted in a Fourier-like manner and the unknown coefficients $\{A_m, B_m\}_{m=0}^{\infty}$ are determined upon application of the boundary condition at Γ_0. However, in this case η_0 is not yet specified. As an alternative we can use the remaining information on the original inner boundary Γ_1 to determine the set of coefficients $\{A_m, B_m\}$. This is achieved by defining a mapping K of η_1 on Γ_1 to η_0 on Γ_0 as is discussed in detail in the original EDEM paper [1]. Those more familiar with the Trefftz method perhaps might recognise the above expression, but instead think of it in terms of the *interior* and *exterior* Trefftz bases for the modified Helmholtz operator with the Γ_2 boundary condition satisfied explicitly. However, it should be remembered that although the solution to (3) is unique, the above expression is not unique and can appear differently as appropriate for different methods.

3. EDEM–Trefftz algorithm

In this section we outline the numerical algorithm for EDEM–Trefftz numerical algorithm for the case of the modified Helmholtz EBVP discussed in Section 2.

Truncating (4) to a finite sum approximation, we have

$$u(r, \theta) = \sum_{m=0}^{M} \left(K_m(\kappa r) - \frac{K_m(\kappa b) I_m(\kappa r)}{I_m(\kappa b)} \right) (A_m \cos(m\theta) + B_m \sin(m\theta)). \tag{5}$$

Consider a finite number, $2M + 1$, of points on the inner boundary, Γ_0. This identifies $2M + 1$ unique points, $\left\{x_j = \left(\theta_j, r\left(\theta_j\right) = t_1\left(\theta_j\right)\right)\right\}_{j=1}^{2M+1}$, on the original boundary Γ_1 corresponding to those points chosen on Γ_0.

Restricting solution (5) to the curve Γ_1, i.e., imposing the condition $u\,|_{\Gamma_1} = \eta_1$ on (5) at those $2M + 1$ points generates $2M + 1$ equations in terms of the unknown coefficients $\{A_m, B_m\}_{m=0}^{M}$ (note $B_0 \equiv 0$), which can be written as the matrix equation $Az = B$, where

$$A = \begin{bmatrix} \alpha_{01} & \alpha_{11} & \cdots & \alpha_{M1} & \beta_{11} & \cdots & \beta_{M1} \\ \alpha_{02} & \alpha_{12} & \cdots & \alpha_{M2} & \beta_{12} & \cdots & \beta_{M2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{02M+1} & \alpha_{12M+1} & \cdots & \alpha_{M2M+1} & \beta_{12M+1} & \cdots & \beta_{M2M+1} \end{bmatrix} \tag{6}$$

with

$$\alpha_{ij} = f_i(r(\theta_j))\cos(i\theta_j), \qquad \beta_{ij} = f_i(r(\theta_j))\sin(i\theta_j),$$

and for non-zero κ, $f_i(r(\theta_j)) = K_i(\kappa r(\theta_j)) - K_i(\kappa b)I_i(\kappa r(\theta_j))/I_i(\kappa b)$ for $i = 1, \ldots, M$, and where

$$z = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_M \\ B_1 \\ \vdots \\ B_M \end{bmatrix}, \qquad B = \begin{bmatrix} \eta_1(r(\theta_1), \theta_1) \\ \eta_1(r(\theta_2), \theta_2) \\ \vdots \\ \eta_1(r(\theta_{2M+1}), \theta_{2M+1}) \end{bmatrix}. \tag{7}$$

The set of unknown coefficients $\{A_m, B_m\}_{m=0}^{M}$ $(B_0 \equiv 0)$ are found by solving this system for the vector z by, say, Gaussian elimination. The solution within the original domain, Ω, is then given by direct application of Eq. (5).

We pause here to remark that the $\kappa \to 0$ limit of (5) coincides with the corresponding solution for the Laplace operator,

$$u(r, \theta) = a_0 \log(r/b) + \sum_{m=1}^{M} (r^{-m} - r^m/b^{2m})(A_m \cos(m\theta) + B_m \sin(m\theta)). \tag{8}$$

Consequently, the solution of (5) for $\kappa = 0.1$ should closely resemble that of the solution for the Laplace equation. A matrix equation similar to (6) and (7) can be derived for the Laplace solution given by (8), where the first column of the matrix A is replaced with logarithmic terms and the Bessel functions in f_i above are replaced by $f_i(r(\theta_j)) = r^{-i}\left(\theta_j\right) - r^{i}\left(\theta_j\right)/b^{2i}$. For more information, see [1].

4. Results

In the remainder of this paper we present the numerical results of the EDEM–Trefftz algorithm for solving two EBVPs on annular domains. All results are obtained by application of the algorithm outlined in Section 3 and are compared against an implementation of the BEM [3–7]. The first example is solved for an annular domain EBVP involving an ellipse for the inner boundary in which the solution of the EDEM–Trefftz algorithm is compared with the BEM for both its numerical accuracy and runtime. The second example is an EBVP on an annular domain with a square inner boundary with similar comparisons presented. The second example is interesting since a problem with Dirichlet, boundary conditions on a square-shaped inner boundary is theoretically unable to be extended (see [1] for more details).

4.0.1. Boundary element method

For comparison, the classical form of the BEM has been used wherein the boundary is represented by linear elements. The value attributed to an element is defined according to a value prescribed at the single node located in the middle of the element. For the modified Helmholtz equation the following Green's fundamental solution,

$$G(r, r') = \frac{K_0(\kappa|r - r'|)}{2\pi}, \tag{9}$$

and gradient

$$\nabla_{r'} G(r, r') = -\frac{K_1(\kappa|r - r'|)}{2\pi}\frac{(r - r')}{|r - r'|}, \tag{10}$$

are used [39,40,10].

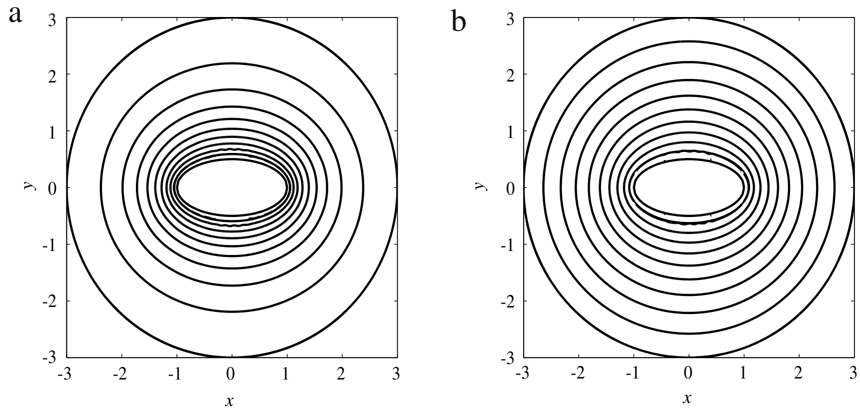

Fig. 2. Contour (iso-potential) plots of the EDEM solution to the Dirichlet problem ((a) $\kappa = 1$, (b) $\kappa = 0.1$) with an elliptical inner boundary. Contours are evaluated in increments of 5 units, with inner contour at $u = 50$ and outer contour at $u = 0$.

4.0.2. Details of the numerical comparison

The numerical solutions of our two EBVPs using both the BEM and EDEM as well as their respective runtimes were obtained using *MATLAB* R2008b on Windows desktop computer with 2.66 GHz Intel Core2 Duo CPU. The differences between the BEM and the EDEM–Trefftz algorithm calculations quoted in the tables were based on the following measure

$$\|v_{\mathrm{EDEM}} - v_{\mathrm{BEM}}\| = \frac{1}{p}\sqrt{\sum_{i=1}^{p}[v_{\mathrm{EDEM}}(\theta_i, r_i) - v_{\mathrm{BEM}}(\theta_i, r_i)]^2}, \tag{11}$$

where v_{EDEM} and v_{BEM} refer to the EDEM and the BEM solution, respectively, evaluated at the discrete interior points (r_i, θ_i) for $i = 1, \dots, p$. Note that we investigate the modified Helmholtz boundary value problem for two values of the Helmholtz parameter, $\kappa = 0.1$ and $\kappa = 1$. We limit the examples to these cases as significantly larger values of κ create numerical difficulties in the evaluation of the modified Bessel functions and thus compromise the results of both methods.

4.1. Example 1: an elliptic inner boundary

Consider first the example of an elliptical inner boundary represented by

$$x^2 + 4y^2 = 1.$$

The outer boundary is a circle of radius $b = 3$ with boundary value $\eta_2 = 0$. The Dirichlet condition on the inner boundary is the constant function $\eta_1 = 50$.

To simplify the calculations, we have exploited the symmetry of the problem in implementing both methods. Instead of solving over the problem's entire domain we solve on the semi-circular region ranging between angles π and 2π. However, we now need to introduce a new zero flux boundary condition across the radial lines defining the extremes of this sub-region. The required derivative, $\partial v/\partial\theta$, derived directly from Eq. (5), will introduce a set of additional equations that can easily be incorporated into the matrix equation for EDEM. The reader may note at this stage that the problem could be further simplified by solving over one quadrant by introducing an extra flux condition. This would also reduce the number of calculations required. However, the chosen simplification will result in only a slight increase in computational cost and is useful for demonstrational purposes as it allows us to observe how EDEM–Trefftz handles symmetry whilst also demonstrating how to simplify for symmetrical domains.

The problem was solved for $\kappa = 0.1$ and $\kappa = 1$ using the EDEM–Trefftz algorithm with 81 points along the inner elliptic boundary and 10 points along each radial boundary. The results are shown in Fig. 2.

Qualitatively, the sequence of curves in Fig. 2 is consistent with the behavior expected of a solution to the modified Helmholtz's equation, with the level curves being more densely distributed as the inner boundary is approached, indicating a higher rate of change closer to that boundary. Note that the inner-most contours take on an elliptic shape and the curves are also symmetric about the *y*-axis as would be expected. For $\kappa = 1$ the level curves are quite dense in the region near the inner elliptic boundary. In the case of $\kappa = 0.1$ the level curves are more evenly distributed throughout the domain as expected since the equation more closely resembles the Laplace equation.

In order to more easily compare the EDEM–Trefftz algorithm results with those of the BEM, we consider snapshots of the solution along a number of "slices" at either constant radius (Fig. 3) or constant angle (Fig. 4). These results are compared with the corresponding results of BEM calculations performed with 200, 400 and 800 elements.

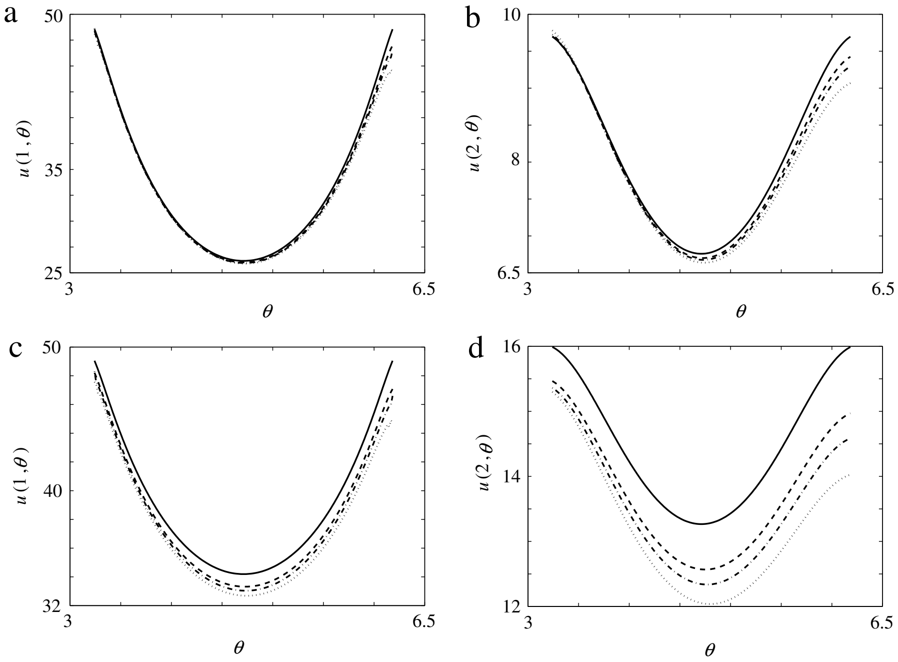

Fig. 3. Constant radius snapshots of solutions to the Dirichlet problem with an elliptical inner boundary. Figure (a) ($\kappa = 1$) (c) ($\kappa = 0.1$) $r = 1$ and (b) ($\kappa = 1$) (d) ($\kappa = 0.1$) $r = 2$, $\theta \in (\pi, 2\pi)$, using the EDEM–Trefftz with 101 points (solid line), and the BEM with 200 (dotted line), 400 (dot-dashed line) and 800 (dashed line) elements.

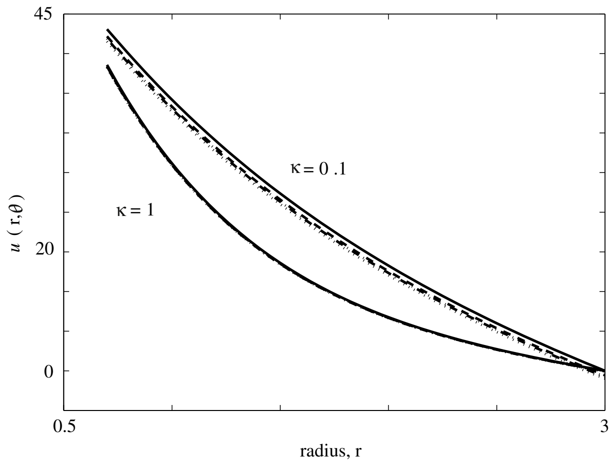

Fig. 4. Constant angle snapshot of solutions to the Dirichlet problem with an elliptical inner boundary for both $\kappa = 0.1$ and $\kappa = 1$ with $\theta = \frac{3\pi}{2}$ and $r \in [0.8, 3]$. The EDEM–Trefftz solution was obtained with 101 points (solid line); BEM solutions were produced with 200 (dotted line), 400 (dot-dashed line) and 800 (dashed line) elements.

It is interesting to note that the BEM does not fully capture the symmetry of the solution as well as the EDEM–Trefftz algorithm (see Fig. 3). This is particularly noticeable when fewer boundary elements are used. In contrast, the new method preserves the symmetry of the problem. The lack of symmetry in the BEM could be a result of the choice of linear elements to represent the boundary. It is possible that more complex geometrical elements would better capture the symmetry of the problem for a lower number of elements. However, this would lead to increased computational complexity and increased runtimes.

Fig. 4 shows the EDEM–Trefftz and BEM solutions at $\theta = 3\pi/2$, plotted as a function of radius. Similar results are found at other angles. Although it is not obvious from Fig. 4, the solutions are very similar close to the inner boundary, with the difference between the solutions increasing closer to the outer boundary (see Fig. 3(a) and (b)). The EDEM–Trefftz clearly has the advantage over the BEM in that it produces more accurate results close to the outer boundary, because the solution is explicitly constructed to satisfy the outer boundary condition on Γ_2; see (4). In the case of the BEM, we are still required

Table 1
Differences between the BEM and EDEM–Trefftz solutions for the Dirichlet problem (modified Helmholtz operator with $\kappa = 1$) with an elliptical inner boundary for various degrees of discretisation, using the measure given by Eq. (11). Values along a row show an increase in the number of boundary points (EDEM); values along a column show an increase in the number of boundary elements (BEM). The total number of points sampled was $p = 4682$.

Number of elements (BEM)	Number of points (EDEM)		
	101	151	201
100	0.010897	0.010905	0.010898
200	0.007331	0.007347	0.007326
400	0.004790	0.004786	0.004797
600	0.003524	0.003532	0.003527
800	0.002909	0.002924	0.002908

Table 2
Differences between the BEM and EDEM–Trefftz solutions for the Dirichlet problem (modified Helmholtz operator with $\kappa = 0.1$) with an elliptical inner boundary for various degrees of discretisation, using the measure given by Eq. (11). Values along a row show an increase in the number of boundary points (EDEM); values along a column show an increase in the number of boundary elements (BEM). The total number of points sampled was $p = 4682$.

Number of elements (BEM)	Number of points (EDEM)	
	101	151
100	0.026913	0.026917
200	0.020874	0.020877
400	0.015437	0.015442
600	0.012956	0.012960
800	0.011472	0.011476

Table 3
Differences between the BEM and EDEM–Trefftz solutions for the Dirichlet problem (Laplace operator) with an elliptical inner boundary for various degrees of discretisation, using the measure given by Eq. (11). Values along a row show an increase in the number of boundary points (EDEM–Trefftz); values along a column show an increase in the number of boundary elements (BEM). The total number of points sampled was $p = 4682$.

Number of elements (BEM)	Number of points (EDEM)		
	101	151	201
100	0.011696	0.011687	0.011684
200	0.007720	0.007713	0.007712
400	0.005473	0.005467	0.005468
600	0.004579	0.004573	0.004574
800	0.004039	0.004034	0.004035

to discretise along the outer boundary and hence the fit is only approximate. This approximate satisfaction of the boundary condition results in small numerical errors in the solution particularly for this simplified implementation of the BEM with a crude element representation.

It is important to note that Figs. 3 and 4 show that for both values of κ the BEM solution approaches the EDEM–Trefftz solution as the number of boundary elements is increased, indicating that the EDEM–Trefftz produces more accurate results for a comparable level of discretisation. It is also interesting to note that the difference between the methods seems to be larger in the case of $\kappa = 0.1$. This is evident particularly in Fig. 3. However, the BEM solution does still appear to approach the EDEM–Trefftz solution, which suggests that the EDEM–Trefftz method handles the modified Helmholtz problem better than the BEM, particularly for small values of κ. Increasing the number of points in the EDEM–Trefftz beyond 101 points does not improve the solution significantly. This fact is summarized in Tables 1–3. The BEM solutions for 100, 200, 400, 600 and 800 elements were summarily compared using the measure given by Eq. (11) against those of the EDEM–Trefftz for 101, 151 and 201 points. This measure is carried out for the modified Helmholtz with $\kappa = 1$ (Table 1) and $\kappa = 0.1$ (Table 2) as well as for the Dirichlet problem with the Laplace operator on the same domain (Table 3). Note that a comparative analysis using a larger number of points using EDEM–Trefftz is not possible in MATLAB as the software does not permit evaluations of higher order modified Bessel functions (for $\kappa = 0.1$, the column for 201 points is omitted for similar reasons). This is not to say that higher order solutions cannot be solved using some other programming environment. However, numerically we are still limited to the finite precision of the numerical program and there will also be an issue on the conditioning of the matrix equation (see later discussion). Despite this limitation, the order for which we can calculate a solution appears to be sufficient for our purposes as the solutions do not appear to need further improvement. Values along a row show the effect of increasing the number of EDEM–Trefftz points for a fixed number of BEM elements. Column values reflect the variation as the number of BEM elements is increased for fixed EDEM–Trefftz number.

We note that the solutions obtained via the two different methods are very similar, even when the discretisation is coarse. Increasing the number of boundary points used in the implementation of the EDEM–Trefftz appears not to improve the comparison. In fact, there is no systematic trend present. In contrast, increasing the number of elements used in the BEM from 100 to 800 reduces the measured difference between the two solution methods by about 57% for $\kappa = 1$, 73% for $\kappa = 0.1$ and 66% for the Laplace equation. The difference is probably the result of the BEM difficulties in solving the

Table 4
Runtime data (s) for the Dirichlet problem (modified Helmholtz operator with $\kappa = 1$) with an elliptical inner boundary solved using the BEM algorithm (second column) as a function of increasing number of elements, and using the EDEM–Trefftz procedure (fourth column) as a function of increasing boundary points. Included is the time taken to evaluate solution values at $p = 5000$ interior points.

Number of BEM elements	Run time (s)	Number of EDEM boundary points	Run time (s)
100	273	101	54
150	451	151	81
200	569	201	109
250	708	251	138

Table 5
Runtime data (s) for the Dirichlet problem (Laplace operator) with an elliptical inner boundary solved using the BEM algorithm (second column) as a function of increasing number of elements, and using the EDEM–Trefftz procedure (fourth column) as a function of increasing boundary points. Included is the time taken to evaluate solution values at $p = 5000$ interior points.

Number of BEM elements	Run time (s)	Number of EDEM boundary points	Run time (s)
100	71	101	0.22
150	107	151	0.34
200	145	201	0.49
250	180	251	0.68

modified Helmholtz equation for low values of κ (See Fig. 3(b), (d) and Fig. 4). This indicates that for a coarse discretisation, the EDEM–Trefftz produces more accurate results than does the BEM. This is consistent with the trends represented by the snapshots in Figs. 3 and 4. Note that although increasing the number of boundary points or elements produces an overall decrease in the difference between solutions, the decrease is not monotonic. The reason for this is unclear and cannot be established from this global comparative measure and likely requires a thorough local study. It is possible that this is the result of minor numerical errors in the BEM approximation. However, since the BEM is not the object of this investigation, we have not pursued the matter further. It is also interesting to note that a larger difference in the measure is obtained for the case of $\kappa = 0.1$ in comparison to the other two cases. However, we still see the trend of decreasing difference as the number of BEM elements increases.

Finally, we compare the runtimes of the two methods for both the modified Helmholtz equation and Laplace equation. We only include the runtimes for $\kappa = 1$ as the runtimes are similar for $\kappa = 0.1$. In determining the runtimes, we include the computational tasks of discretising the boundary, generating and solving the respective matrix equations and calculating $p = 5000$ interior points. The results are presented in Tables 4 and 5.

There is a significant difference between the runtimes of the two methods, with the EDEM–Trefftz running faster than the BEM for a comparable discretisation in all cases. In the modified Helmholtz case, the EDEM runtimes are about one fifth of the BEM, while for the Laplace operator the EDEM–Trefftz runs significantly quicker (i.e., two orders of magnitude). As both Dirichlet problems are solved on exactly the same domain, the increase in runtime for the modified Helmholtz is due to the type of functions being calculated rather than any other complexity. In the case of the modified Helmholtz operator, the solution requires calculation of modified Bessel functions which result in increased computational costs. A large proportion of the runtimes quoted can be attributed to the evaluation of solution values inside the domain. This is true of both methods. However, even without this task the EDEM–Trefftz still runs significantly quicker than the BEM. It should be noted that as the BEM requires explicit evaluation of boundary integrals, the large BEM runtimes are certainly attributed to this task and a further consequence of the use of Gaussian quadrature. With the modified Helmholtz operator it is possible to affect these integrations analytically [3], which would potentially speed up the BEM operations and improve the comparison. However, this is not generally possible for other elliptic operators, which suggests that the comparison made here better reflects the more general situation.

It is interesting to note that while the runtimes for the BEM appear to increase linearly with the number of elements, the EDEM–Trefftz runtimes increase more rapidly. Extrapolating to very fine meshes, it may be that the BEM will outperform the EDEM–Trefftz, but this is likely to occur for discretisations that exceed the current limits of computation. Also, this issue seems to be offset by the EDEM–Trefftz's ability to obtain what appear to be very accurate solutions for a low number of points.

Finally, we should also comment on the behavior of the method with respect to the variable κ. In the above examples and the next, κ has only been considered within the range (0, 1]. There is however the question about the behavior of the EDEM–Trefftz method for cases where $\kappa \gg 1$. Increased values of κ result in a marked increase in rate of change in the solution close to the inner boundary. Solutions obtained from EDEM–Trefftz show that the method can effectively handle BVPs with κ values up to 50 in the cases tested. The latter results show behavior consistent with those discussed above in terms of the method's accuracy and runtime. However, in the case of an extreme value of κ, e.g., $\kappa = 100$, the solution can suffer from large numerical errors, particularly close to the inner boundary, Γ_1. The large value of κ effectively leads to what is known in the fluid dynamics and potential field literature as boundary layer phenomena. Here, the effect of a non-zero boundary source is confined to a very narrow region near that boundary of the domain, of thickness of order $\kappa^{-1} = 0.01$. Only for BVPs with elliptical boundaries closely resembling a circle is the EDEM–Trefftz method able to capture

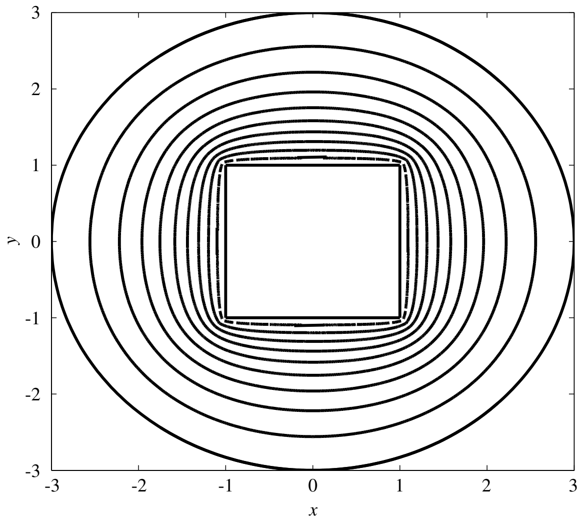

Fig. 5. Contour (iso-potential) plots of the EDEM–Trefftz solution to the Dirichlet problem with a square inner boundary. Contours are evaluated in increments of 5 units, with inner contour at $u = 50$ and outer contour at $u = 0$.

this behavior effectively. However, it should be noted that this problem is not just confined to EDEM–Trefftz, the phenomena is also difficult to capture using other numerical schemes. The implementation of Shankar's least square method discussed in the introduction and results obtained using the BEM have also been tested for these extreme cases and were also found to be deficient.

4.2. Example 2: a square inner boundary

As a second example, we consider the Dirichlet problem with a square inner boundary with constant boundary data but this time for $\kappa = 1$ only. The reason for this choice is to study the influence of boundary corners on performance. In contrast to the previous example of a smooth boundary, the piece-wise smooth example of a square boundary should be a more demanding test of the present numerical approach to EDEM–Trefftz. We note that this combination of boundary shape and boundary data invalidates the definition of the operator, K, which maps the original boundary conditions on Γ_1 to the new boundary conditions on the new boundary of the extended domain, Γ_0. In [1] we established in the case of the Laplace operator sufficient and necessary conditions for extendability of the boundary data, $\eta_1(\theta)$. This analysis was based on the condition that the boundary Γ_1 (i.e., $t_1(\theta)$) was piece-wise smooth. Although the corresponding analysis for more general elliptic partial differential operators has not been presented, we anticipate that the arguments and results would carry over to the modified Helmholtz operator. In this case, the present problem of a piece-wise smooth square boundary with constant boundary data would not meet the extendability criteria and hence a numerical solution would therefore not be possible. Remarkably, with a judicious choice in the numerical implementation, EDEM can still be used to find a solution to this problem, a fact that does not seem to have been remarked upon in the literature to date.

The square is centered on the origin with side length 2 and boundary value $\eta_1 = 50$. The outer boundary is again a circle of radius $r = 3$, centered on the origin with boundary value $\eta_2 = 0$. Once again, we shall exploit the symmetry of the problem and focus on the sub-domain between angles $5\pi/4$ and $7\pi/4$, with the same zero flux conditions applied on the radial lines defining the extremes of this region. The problem is solved using the EDEM–Trefftz algorithm with 81 inner boundary points and 10 points along each radial line. The solution is shown in Fig. 5.

Once again, the EDEM–Trefftz results appear to be qualitatively consistent with the behavior expected of the solution of the modified Helmholtz equation, with the inner-most level curves being densely distributed and taking on the square shape of the inner boundary. Graphical comparisons between the EDEM–Trefftz results and the BEM results at radial and angular snapshots are shown in Figs. 6 and 7, respectively.

Figs. 6 and 7 show similar results to those of the previous example, although in this case the BEM solution more closely matches the solution obtained using EDEM–Trefftz. The EDEM solution, however, still better reflects the symmetry of the problem. Overall, the BEM solution approaches the EDEM–Trefftz solution as the number of boundary elements is increased. A summary comparison using the measure given by Eq. (11) is presented in Table 6.

The data supports the quantitative findings of the previous example. Increasing the number of boundary points used in the EDEM–Trefftz does not significantly affect the quality of the EDEM–Trefftz solution. On the other hand, increasing the number of boundary elements used in the BEM decreases the difference between the two solutions. This again suggests that the EDEM–Trefftz produces more accurate results than does the BEM, particularly when the discretisation is coarse.

Finally, the runtimes for the two methods are very nearly the same as found with the previous example of a smooth inner boundary (and consequently are not included here). We attribute some very slight increases in runtimes to the more

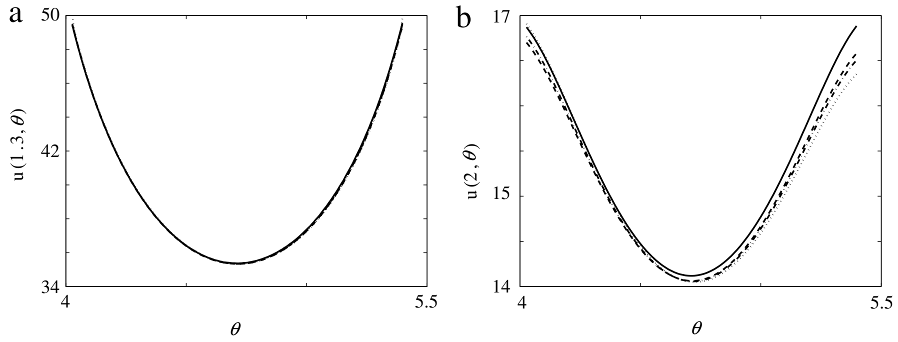

Fig. 6. Constant radius snapshots of solutions to the Dirichlet problem with a square inner boundary. Figure (a) $r = 1.3$ with $\theta \in (4.2827, 5.1421)$ and (b) $r = 2, \theta \in \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$, using the EDEM–Trefftz with 101 points (solid line), and the BEM with 200 (dotted line), 400 (dot-dashed line) and 800 (dashed line) elements.

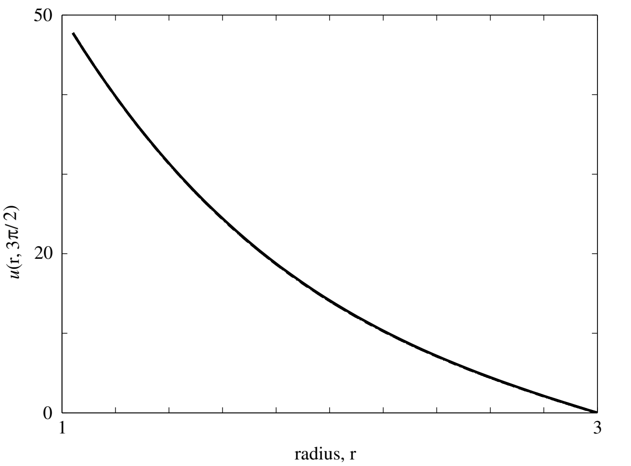

Fig. 7. Constant angle snapshot of solutions to the Dirichlet problem with a square inner boundary with $\theta = \frac{3\pi}{2}$ and $r \in [0.1, 3]$. The EDEM–Trefftz solution was obtained with 101 points (solid line); BEM solutions were produced with 200 (dotted line), 400 (dot-dashed line) and 800 (dashed line) elements.

Table 6
Differences between the BEM and EDEM–Trefftz solutions for the Dirichlet problem with a square inner boundary for various degrees of discretisation, using the measure given by Eq. (11). Values along a row show an increase in the number of boundary points (EDEM); values along a column show an increase in the number of boundary elements (BEM). The total number of points sampled was $p = 4896$.

Number of elements (BEM)	Number of points (EDEM)		
	101	151	201
100	0.010261	0.010262	0.010325
200	0.004646	0.004633	0.004646
400	0.003505	0.003477	0.003432
600	0.002634	0.002598	0.002527
800	0.002502	0.002467	0.002383

complex pre-processing phase wherein the discretisation takes place. Overall, the runtimes beyond the pre-processing stage appear to be independent of the problem geometry for the same number of points.

4.3. Ill-conditioning

The reader at this stage may be wondering about the behavior of the A matrix (6) in the EDEM–Trefftz algorithm. The matrix is ill-conditioned as a result of its direct construction, which can be seen via its condition number (the ratio between the largest and smallest values of the singular decomposition). For the EDEM–Trefftz algorithm the condition number will vary depending on the level of discretisation used (number of boundary points) and the parameter κ. In Table 7 we calculate the condition number for the elliptic inner boundary problem for a variety of discretisations and levels of κ. Note that for the case $\kappa = 0.1$ and $M = 201$, a solution was not calculated due to the precision issue discussed earlier.

Table 7
Condition number of matrix A of EDEM–Trefftz for ellipse inner boundary. Values along a row show an increase in κ; values along a column show an increase in the number of boundary points M (EDEM).

κ	M Number of points (EDEM)			
	51	101	151	201
0.1	2.56E+103	2.93E+194	8.22E+297	N/A
1	3.54E+67	3.74E+133	6.34E+211	1.53E+293
10	1.13E+43	1.48E+77	1.79E+127	1.01E+180
50	5.64E+70	3.68E+81	1.03E+100	2.60E+129
100	7.54E+116	3.11E+122	3.35E+132	9.69E+148

The results for the square inner boundary case has not been included as they behave in a similar manner to those of the ellipse problem. As can be seen, even for a rough discretisation (e.g., 51 boundary points) the condition number of the matrix is of a significant order and clearly shows the system to be ill-conditioned. As the order M of (5) is increased the condition number of the matrix A also increases due to inclusion of higher order modified Bessel functions in the solution. This results in a significant increase in the difference between the smallest and largest elements of the matrix and hence compounds the ill-conditioned state of the matrix. Similarly, for the case of large and small κ, a large condition number is also obtained. In this case, a large condition number is the result of evaluations of modified Bessel functions. That is, the elements are either large in magnitude or close to zero. The largest condition number occurs for small κ and a large discretisation parameter M.

For reasons of space we defer any further discussion on the ill-posed behavior of the EDEM–Trefftz algorithm. However, the issue of ill-posedness of EDEM is an important one warranting serious consideration. It is the subject of a separate paper. It should be noted though, that no attempt has been made in this paper by the authors to regularise this method to account for the ill-posedness of the method or ill-conditioning of the resulting matrix.

5. Summary remarks

The necessity of providing efficient algorithms for the solution of linear partial differential equations for systems involving complicated boundaries is without dispute, since analytical methods of solution are often restricted to domains that have little or no departure from simple geometries. Traditional numerical approaches are not without their disadvantages. The Boundary Element Method, Finite Element Method, Finite Difference Method and Boundary Point Method can be tricky to implement for a non-numerically trained scientist, they have slow runtimes and give little direct insight into the problem beyond the numerical data they output. The Extended-Domain-Eigenfunction-Method proposed in [1] provides an alternative to these methods. By extending a complicated annular-like domain to one which has a simple geometry, the classic eigenfunction solution can be obtained. By restricting this solution to the original domain, we can obtain the solution to the original problem.

In this paper a numerical algorithm was proposed which overlaps with the Trefftz method. This algorithm was applied to two EBVPs on 2D annular domains for the specific case of the modified Helmholtz operator, which has not been extensively studied, and the results compared against those of a simple linear element implementation of the BEM. The two examples support the notion that the EDEM–Trefftz numerical implementation is able to produce accurate estimates of solutions to EBVPs, consistent with those found using the BEM, but does so much more rapidly (at least for the Laplace and modified Helmholtz operator). It seems that these solutions are even more accurate than those produced using the simplest variant of the BEM (i.e., linear elements with a single node point). It is possible (and likely) that greater accuracy may be achieved by the BEM through the use of more complex geometric elements and greater number of node points, but this of course would necessarily result in increased computational effort and computation time. The EDEM–Trefftz algorithm, as implemented here, seems to achieve high numerical accuracy, without sacrificing computation time.

Finally, we remark that our work addresses some questions left open in Shankar's paper [18]. For example, our comparison with the Boundary Element Method answers the question of how well an approach such as EDEM compares with other numerical methods. Also, Shankar's questions on the validity and ill-posedness of the theoretical approach, and the ill-conditioned nature of the associated matrix problem, are clarified in this and our previous paper [1], where we have investigated the particular cases of the Laplace and Helmholtz operators. These investigations will be extended to more general circumstances in future work.

New approaches offering these sort of improvements in accuracy and speed are of growing interest to the animation and computer graphic industries to produce more realistic special effects as well as to the broader scientific community to increase our scientific understanding of physical phenomena. The results obtained above suggest EDEM and Trefftz based methods could be of potential use in these fields and will undoubtedly be a subject for future research.

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