A GENERAL ALGORITHM FOR THE NUMERICAL EVALUATION OF NEARLY SINGULAR BOUNDARY INTEGRALS IN THE EQUIVALENT NON-SINGULAR BIES WITH INDIRECT UNKNOWNS

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ABSTRACT

The accurate evaluation of nearly singular boundary integrals is an important issue in boundary element analysis, and the importance of this problem is next to the singular boundary integrals. Although many ways of evaluating nearly singular integrals have been developed in recent years, and obtained varying degree of success, questions still remain. In this paper, a new efficient transformation is proposed, based on a new idea of transformation with variables. The proposed transformation can remove the near singularity efficiently by smoothing out the rapid variations of the integrand of nearly singular integrals, and improve the accuracy of numerical results of nearly singular integrals greatly without increasing the computational effort. Numerical examples of potential problems are given, with results, showing the high efficiency and the stability of the suggested approach, even when the internal point is very close to the boundary.

Key Words: potential problem, boundary layer effect, nearly singular integrals, transformation.

I. INTRODUCTION

The accurate computation of the kernels' integration is a critical aspect in the implementation of the boundary element method (Kisu *et al*., 1988; Chen *et al*., 1994). These kernels are weakly singular, strongly singular, or even hypersingular functions when the collocation point belongs to the integration element, and many effective methods(Brebbia *et al*., 1984; Kisu *et al*., 1988; Sun., 1999; Liu, 2000; Chen H.B. *et al*., 2001; Chen J. T. *et al*., 1994;1999; 2004; 2005; 2006a; 2006b; 2007a;2007b; Niu *et al*., 2004; Zhang *et al*., 2000; 2001; 2004; 2006; Wang, 2005) have been developed to handle them. Another important issue is the integration of the kernels for the collocation points which are close to but not on the integration element. The ensuing integrals, although regular in nature, are termed nearly weak singular, nearly strong singular and nearly hypersingular integrals since the integrand varies very rapidly within the integration interval, the integrand varies more rapidly the closer the collocation point is to the integration element, and can not be computed accurately with standard Gaussian quadrature. Practice shows that using standard Gaussian quadrature procedures, which neglect the pathological behavior of the integrand as the collocation point approaches the integration element, will lead to a decrease in the accuracy of the solution, even to no accuracy, which is the so-called boundary layer effect.

The accurate numerical evaluation of the nearly singular boundary integrals is crucial to some engineering problems, such as crack problems when the crack tip is deformed to have a small opening displacement, contact problems when the contact area of the two contacting bodies is very small, as well as thin or shell-like structure problems.

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In BEM, the importance of the accurate evaluation of nearly singular integrals is next in importance to singular boundary integrals. Therefore, a great amount of attention has been attracted and many numerical techniques have been developed for it in recent years. These proposed methods can be divided on the whole into two categories: "global algorithms" (Cruse, 1974; J.C. Lachat, J.O, 1976; Wang *et al*., 1994; Chen *et al*., 1998; Zhang *et al*., 2000; Mukerjee *et al*., 2000; Sladek *et al*., 1993; Granados, 2001) and "local algorithms" (Jun, 1985; Telles, 1987; Cerrolaza *et al*., 1989; Tanaka *et al*., 1991; Lutz, 1992; Cruse, *et al*., 1993; Lutz, 1993; Huang *et al*, 1993; Luo *et al*., 1998; Schulz *et al*., 1998; Liu, 1998; Johnston, 1999; Sladek *et al*., 2000; Niu *et al*., 2000; 2001; 2005; Chen *et al*., 2001; Zhang *et al*., 2001; Padhi *et al*., 2001; Ma, 2001; 2002a; 2002b; Zhou *et al*., 2003; MA *et al*., 2004; Zhang X.S. *et al*., 2004; Davey *et al*., 2004; Wang *et al*., 2005;). The "global algorithms" are mainly to calculate indirectly or avoid calculating the nearly singular integrals by establishing new regularized boundary integral equations, such as the virtual boundary element method (Sun *et al*., 1999; Zhang *et al*., 2001), rigid-body displacement method, or the simple solution method, as well as other global regularization methods (Cruse, 1974; Lachat, 1976; Sladek *et al*., 1993; Wang *et al*., 1994; Chen *et al*., 1998; 2001; Liu, 2000; Mukerjee *et al*., 2000; Granados, 2001; Zhang Y.M. *et al*., 2004). The virtual boundary element method can avoid calculating the singular integrals or nearly singular integrals, but its theoretical foundation is only confirmed in some cases(Sun *et al*., 1999; Zhang *et al*., 2001). Simple solution method and rigid-body displacement method, which benefit from the regularization ideas and techniques for the singular integrals, eliminate the nearly zero factor in the denominator of kernel function by the zero factor in density function, but the accuracy of their calculated results are not satisfactory. The "local algorithms" are calculating or approximating the nearly singular integrals directly. They usually include interval subdivision(Jun *et al*., 1985; Tanaka *et al*., 1991), special Gaussian quadrature (Lutz, 1992; MA *et al*., 2004), analytical or semi-analytical methods (Cruse *et al*., 1993; Schulz *et al*., 1998; Liu, 1998; Zhang *et al*., 2001; Padhi *et al*., 2001; Niu *et al*., 2001; 2005; Zhou *et al*., 2003; Zhang X.X. *et al*., 2004; Davey, 2004; Wang *et al*., 2005) and transformations (Telles, 1987; Cerrolaza, 1989; Lutz, 1993; Huang, 1993; Luo *et al*., 1998; Johnston, 1999; 2000; Sladek *et al*., 2000; Ma *et al*, 2001; 2002a; 2002b; Zhang *et al*, 2006), etc. The interval subdivision method is an effective method but it is not recommended, because the number of the divided intervals depends on the distance from the computing point to the boundary, with more intervals, the closer the computing point is to the boundary, which requires great computation efforts and can increase the accumulative error. The special Gaussian quadrature method needs to determine the weight coefficients and integration points of the quadrature formula based on the form of the integrand, and the weight coefficients and integration points will change with the changes of the distance from the calculating point to the boundary curve. Thus, heavy and complicated derivation work must be done in this method and it usually is used in conjunction with other methods. The analytical method for nearly singular integrals seems more difficult than for singular integrals and is generally considered impossible for curved elements. The semi-analytical method primarily separates the nearly singular parts by the "plus-minus method" with the separated parts computed analytically and the regular parts computed by standard Gaussian quadrature formula. However, this method does not remove the near singularity completely, and the regular parts still retain weak near singularity, meanwhile, more derivation work has to be done before numerical implementation. At present, the most common of the "local algorithms" are various non-linear transformations, for example, the cubic polynomial transformation (Telles, 1987), the bi-cubic transformation (Cerrolaza, 1989), the sigmoidal transformation (Johnston, 1999), the semi-sigmoidal transformation (Johnston, 2000), the coordinate optimization transformation (Sladek *et al*., 2000), the attenuation mapping method (Lutz, 1993; Luo *et al*., 1998), the rational transformation (Huang *et al*., 1993), and the distance transformation (Ma *et al*., 2001; 2002a; 2002b).

The basic ideas of the above transformations can be generalized into two categories: one (Telles, 1987; Cerrolaza, 1989; Lutz, 1993; Luo *et al*., 1998; Johnston, 1999; 2000; Sladek *et al*., 2000) is removing the nearly zero factor in the denominator of the kernels using zero factor (usually generated by Jacobian); the other one (Huang *et al*., 1993; Ma *et al*., 2001; 2002a; 2002b) is converting the nearly zero factor in the denominator of the kernels to be part of the numerator, which profits from the idea of the reciprocal transformation for the regularization of weakly singular integrals. Numerical tests show that the transformations based on the former idea are effective for the calculation of weakly singular integrals but not satisfactory for strong singular or hypersingular integrals. The latter transformations, based on the idea of reciprocal transformation, can convert nearly singular kernels into normal kernels, but the original regular parts behave nearly singularly after the transformations, so they are suitable only for a case when the regular part of the integrand is constant.

Recently, some new developments (Shiah *et al*., 2006; 2007; Chen, 2007) in the range of nearly singular computation have been reached.

This paper constructs a general variable transformation based on the idea of diminishing the difference of the orders of magnitude or the scale of change of operational factors. The constructed transformation can smooth out the rapid variations of nearly singular kernels and extremely high accuracy of numerical results for nearly singular integrals can be achieved with this method. Furthermore, the suggested transformation is effective whatever the regular part of the integrand may be.

II. THE NEARLY SINGULAR INTEGRALS IN THE EQUIVALENT NONSINGULAR BIES

In this paper, we always assume that Ω is a bounded domain in $\boldsymbol{R}^2$, Ω^c its open complement, and $\Gamma = \partial\Omega$ their common boundary. $\boldsymbol{t}(\boldsymbol{x})$, $\boldsymbol{n}(\boldsymbol{x})$ are the unit tangent and outward normal vectors of Γ to domain Ω at point $\boldsymbol{x}$, respectively. In BEM, if boundary conditions are given properly, the domain variables can be computed by corresponding integral equations after the boundary quantities are all obtained. For plane potential problems, the equivalent non-singular BIEs with indirect variables are given in Zhang (2006). For the domain Ω, the equations are given as

$$\int_\Gamma \phi(\boldsymbol{x})d\Gamma_x + \int_\Omega f(\boldsymbol{x})d\Omega = 0 \quad (1)$$

$$u(\boldsymbol{y}) = \int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x + \int_\Omega f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega + C, \quad \boldsymbol{y}\in\Gamma \quad (2)$$

$$\begin{aligned}\nabla u(\boldsymbol{y}) = \phi(\boldsymbol{y}) &+ \int_\Gamma [\phi(\boldsymbol{x})-\phi(\boldsymbol{y})]\nabla u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma \\ &- \phi(\boldsymbol{y})\left\{\int_\Gamma [\boldsymbol{t}(\boldsymbol{x})-\boldsymbol{t}(\boldsymbol{y})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{t}_x}d\Gamma + \int_\Gamma [\boldsymbol{n}(\boldsymbol{x})-\boldsymbol{n}(\boldsymbol{y})]\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}_x}d\Gamma\right\} \\ &+ \int_\Omega f(\boldsymbol{x})\nabla u^*(\boldsymbol{x},\boldsymbol{y})d\Omega, \quad \boldsymbol{y}\in\Gamma. \end{aligned} \quad (3)$$

For the domain Ω^c, the equations are given as

$$\int_\Gamma \phi(\boldsymbol{x})d\Gamma_x + \int_{\Omega_c} f(\boldsymbol{x})d\Omega^c = 0 \quad (4)$$

$$u(\boldsymbol{y}) = \int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x + \int_{\Omega^c} f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Omega^c + C, \quad \boldsymbol{y}\in\Gamma. \quad (5)$$

$$\begin{aligned}\nabla_y u(\boldsymbol{y}) = &\int_\Gamma [\phi(\boldsymbol{x})-\phi(\boldsymbol{y})]\nabla_y u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x \\ &- \phi(\boldsymbol{y})\left\{\int_\Gamma [\boldsymbol{t}^c(\boldsymbol{x})-\boldsymbol{t}^c(\boldsymbol{y})]\nabla u^*(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{t}^c(\boldsymbol{x})d\Gamma_x + \int_\Gamma [\boldsymbol{n}^c(\boldsymbol{x})-\boldsymbol{n}^c(\boldsymbol{y})]\nabla u^*(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}^c(\boldsymbol{x})d\Gamma_x\right\} \\ &+ \int_{\Omega^c} f(\boldsymbol{x})\nabla_y u^*(\boldsymbol{x},\boldsymbol{y})d\Omega^c, \quad \boldsymbol{y}\in\Gamma, \end{aligned} \quad (6)$$

Where $\boldsymbol{t}^c(\boldsymbol{x})$, $\boldsymbol{n}^c(\boldsymbol{x})$ are the unit tangent and outward normal vectors of Γ to domain Ω^c at point $\boldsymbol{x}$, respectively.

For the internal point $\boldsymbol{y}$, the integral equations can then be written as

$$u(\boldsymbol{y}) = \int_\Gamma \phi(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x + \int_{\hat{\Omega}} f(\boldsymbol{x})u^*(\boldsymbol{x},\boldsymbol{y})d\hat{\Omega} + C, \quad \boldsymbol{y}\in\hat{\Omega} \quad (7)$$

$$\nabla_y u(\boldsymbol{y}) = \int_\Gamma \phi(\boldsymbol{x})\nabla_y u^*(\boldsymbol{x},\boldsymbol{y})d\Gamma_x + \int_{\hat{\Omega}} f(\boldsymbol{x})\nabla_y u^*(\boldsymbol{x},\boldsymbol{y})d\hat{\Omega}, \quad \boldsymbol{y}\in\hat{\Omega}. \quad (8)$$

In the Eqs. (1)-(8), $\phi(\boldsymbol{x})$ is the density function to be determined, $f(\boldsymbol{x})$ the body function, and in Eq. (7) and (8) $\hat{\Omega} = \Omega$ or Ω^c.

For the Eqs. (7) and (8), when the internal point $\boldsymbol{y}$ is far enough from the boundary Γ, the standard Gaussian quadrature procedures without any transformation can get desired results. However, when the point y approaches the boundary Γ, the accuracy of the results computed by the conventional quadrature algorithm degenerates rapidly, and with the internal point closer to the boundary, the results will be out of true, namely, the "boundary layer effect" because of the near singularity of the kernel of the fundamental solution. These nearly singular integrals can be expressed as

$$\begin{cases} I_1 = \int_\Gamma \psi(\boldsymbol{x})\ln r^2 d\Gamma \\ I_2 = \int_\Gamma \psi(\boldsymbol{x})\frac{1}{r^{2\alpha}}d\Gamma \end{cases}, \quad (9)$$

where $\alpha > 0$, r the distance between the internal point $\boldsymbol{y}$ and the integration point $\boldsymbol{x}$, and $\psi(\boldsymbol{x})$ is a well-behaved function (which may be different in different integrals). Obviously, the near singularities of the integrals in Eq. (9) come from the distance r.

1. Linear Element Approximation of Geometry Boundary Γ

Assuming $\boldsymbol{x}^1 = (x_1^1, x_2^1)$, $\boldsymbol{x}^2 = (x_1^2, x_2^2)$ are the two extreme points of the linear element Γ_j, then the element Γ_j can be expressed as

$$x_k(\xi) = N_1(\xi)x_k^1 + N_2(\xi)x_k^2 \ (-1 \le \xi \le 1),\ k = 1, 2 \tag{10}$$

here $N_1(\xi) = \frac{1}{2}(1 - \xi)$, $N_2(\xi) = \frac{1}{2}(1 + \xi)$. Letting $s_i = x_i^2 - x_i^1$, $w_i = y_i - \frac{1}{2}(x_i^2 + x_i^1)$, one has

$$r_{,i} = \frac{r_i}{r} = \frac{y_i - x_i}{r} = \frac{s_i\xi/2 + w_i}{r} \tag{11}$$

$$r^2 = |\boldsymbol{x} - \boldsymbol{y}|^2 = r_i r_i = A\xi^2 + B\xi + E = L^2[(\xi - \eta)^2 + d^2], \tag{12}$$

where $A = \frac{1}{4}s_i s_i$, $B = s_i w_i$, $E = w_i w_i$, $\eta = -\frac{B}{2A}$, $L = \sqrt{A}$, $d = \frac{1}{2A}\sqrt{4AE - B^2}$. If $\sqrt{E} < \sqrt{A}$, i.e., when the distance from the point $\boldsymbol{y}$ to the middle point of the element Γ_j is less than half length of the element, by using the hölder inequality we have

$$|\eta| = \left|\frac{s_i w_i}{2A}\right| \le \left|\frac{\sqrt{s_i s_i}\sqrt{w_i w_i}}{2A}\right| = \left|\frac{\sqrt{E}}{\sqrt{A}}\right| < 1 .$$

Thus, the integrals I_1 and I_2 in Eq. (9) can respectively be divided into two parts at point η as follows

$$I_1 = \ln L^2 \int_{-1}^{1} g(\xi)d\xi + \left\{\int_{-1}^{\eta} + \int_{\eta}^{1}\right\} g(\xi)\ln[(\xi - \eta)^2 + d^2]d\xi \tag{13}$$

$$I_2 = \frac{1}{L^{2\alpha}}\left\{\int_{-1}^{\eta} + \int_{\eta}^{1}\right\} \frac{g(\xi)}{[(\xi - \eta)^2 + d^2]^{\alpha}}d\xi , \tag{14}$$

where $g(\cdot)$ is the regular function that consists of shape function and Jacobian ($r_i = y_i - x_i$ included in the Eq. (14)).

2. "Arc Element" Approximation of the Boundary Γ

The interpolation approximation with "arc element", which was proposed in Zhang (2004), is nearly accurate if the boundary of the domain is circular. Suppose (R, θ_1), (R, θ_2) are the coordinates of the two extreme points of the arc element Γ_j, respectively, then the element Γ_j can be expressed as

$$\begin{cases} x_1 = R\cos\theta \\ x_2 = R\sin\theta \end{cases},\ \theta = \frac{1 - \xi}{2}\theta_1 + \frac{1 + \xi}{2}\theta_2 \ (-1 \le \xi \le 1)$$

For the interior point $\boldsymbol{y} = (R_0\cos\theta_0, R_0\sin\theta_0)$, we can suppose $\theta_1 < \theta_0 < \theta_2$, then $\theta_0 = \frac{1 - \eta}{2}\theta_1 + \frac{1 + \eta}{2}\theta_2 (-1 < \eta < 1)$. Thus

$$r^2 = |\boldsymbol{x} - \boldsymbol{y}|^2 = 4RR_0\{\sin_2[\beta(\xi - \eta)] + d^2\},$$

here $\gamma = \beta(\xi - \eta)$, $\beta = \frac{\theta_2 - \theta_1}{4}$, $d = \frac{R - R_0}{4\sqrt{RR_0}}$.

The integrals I_1 and I_2 in Eq. (9) can respectively be divided into two parts at poin η there as follows

$$I_1 = \int_{-1}^{1} g(\xi)\ln(4RR_0)d\xi + \left\{\int_{-1}^{\eta} + \int_{\eta}^{1}\right\} g(\xi) \cdot \ln\{\sin^2[\beta(\xi - \eta)] + d^2\}d\xi \tag{15}$$

$$I_2 = \frac{1}{L^{2\alpha}}\left\{\int_{-1}^{\eta} + \int_{\eta}^{1}\right\} g(\xi)\frac{1}{\{\sin^2[\beta(\xi - \eta)] + d^2\}^{\alpha}}d\xi , \tag{16}$$

Where $L = 2\sqrt{RR_0}$ and $g(\cdot)$ is the regular function that consists of shape function and Jacobian.

III. THE TRANSFORMATION FOR THE NEARLY SINGULAR INTEGRALS

The variable in the two integrals in front on the right side of Eq. (13) can be replaced by

$$\xi = \eta - k_1 d(e^{k(1 + t)} - 1),\ \xi = \eta + k_2 d(e^{k(1 + t)} - 1) \tag{17}$$

Here $k_1 = 1 + \eta$, $k_2 = 1 - \eta$, $k = [\ln(1 + d) - \ln d]/2$. Then we have

$$\begin{aligned} I_1 = {} & \ln L^2 \int_{-1}^{1} g(\xi)d\xi + kd\ln d^2 \int_{-1}^{1} [k_1 g_1(t) + k_2 g_2(t)]e^{k(1 + t)}dt \\ & + kk_1 d\int_{-1}^{1} g_1(t)e^{k(1 + t)}\ln[k_1^2(e^{k(1 + t)} - 1)^2 + 1]dt \\ & + kk_2 d\int_{-1}^{1} g_2(t)e^{k(1 + t)}\ln[k_2^2(e^{k(1 + t)} - 1)^2 + 1]dt , \end{aligned} \tag{18}$$

Where $g_1(t) = g[\eta - k_1 d(e^{k(1 + t)} - 1]$, $g_2(t) = g[\eta + k_2 d(e^{k(1 + t)} - 1)]$.

The two integrals on the right side of Eq. (14) can be transformed in the same way as Eq. (17) into the following forms

$$I_2 = \frac{kk_1 d^{1 - 2\alpha}}{L^{2\alpha}}\int_{-1}^{1} \frac{g_1(t)e^{k(1 + t)}}{[k_1^2(e^{k(1 + t)} - 1)^2 + 1]^{\alpha}}dt + \frac{kk_2 d^{1 - 2\alpha}}{L^{2\alpha}}\int_{-1}^{1} \frac{g_2(t)e^{k(1 + t)}}{[k_2^2(e^{k(1 + t)} - 1)^2 + 1]^{\alpha}}dt \tag{19}$$

here $g_1(t)$, $g_2(t)$ are the same as those above.

For the Eqs. (15) and (16), we have

$$I_1 = \ln(4RR_0)\int_{-1}^{1} g(\xi)d\xi + \int_0^{t_1} g_1(t)\ln(t^2+d^2)dt + \int_0^{t_2} g_2(t)\ln(t^2+d^2)dt \quad (20)$$

$$I_2 = \frac{1}{h}\left[\int_0^{t_1} \frac{g_1(t)}{(t^2+d^2)^{\alpha}}dt + \int_0^{t_2} \frac{g_2(t)}{(t^2+d^2)^{\alpha}}dt\right], \quad (21)$$

where

$$h = L^{2a},\ t_1 = \sin[\beta(1+\eta)],\ t_2 = \sin[\beta(1-\eta)],$$

$$g_1(t) = \frac{1}{\beta\sqrt{1-t^2}}g(\eta - \frac{1}{\beta}\arcsin t),$$

$$g_2(t) = \frac{1}{\beta\sqrt{1-t^2}}g(\eta + \frac{1}{\beta}\arcsin t).$$

As long as $\theta_2 - \theta_1 \le \frac{\pi}{4}$, so $|\beta(1 \pm \eta)| < \frac{\pi}{8}$, hence

$$|t_1|,\ |t_2| = |\sin[\beta(1 \pm \eta)]| \le \sin\frac{\pi}{8} \approx 3.826834\text{E-01} << 1$$

thus $g_1(t)$, $g_2(t)$ are both regular functions.

Similarly, the second and the third integrals on the right hand side of the Eq. (20) and the integrals in the Eq. (21) also can be transformed in the same way as Eq. (17).

IV. NUMERICAL EXAMPLES

In this section, the transformation suggested for evaluating the nearly singular integrals is applied to determine the potentials and gradients of plane problems in the BEM. In the following three examples the potentials or the potential gradients are computed using EBIEs, in which the above types of integrals in Eq. (9) are calculated by the developed algorithm at internal points increasingly close to the boundary. The results obtained by using the present method, the ones computed by the conventional algorithm and exact solutions are all presented for convenience of comparison, which can demonstrate the usefulness of the proposed method. Eight-point Gaussian integration is used unless specified explicitly when the numerical integrals are done.

Example 1. A prism with square section and infinite length is considered in the example with prescribed temperatures or fluxes on boundaries, as shown in Fig. 1.

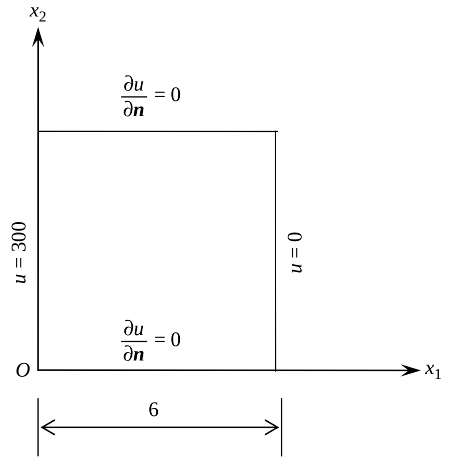

Fig. 1 The boundary conditions on the square section of the prism

There are 28 uniformly linear boundary elements divided over the boundary of the square section, and linear discontinuous interpolation (Zhang, 2004) is adopted to approximate the boundary functions.

The temperatures u and the fluxes q_{x_1} (in the x_1 -direction) of internal points are computed with their boundary integral equations in BEM. The conventional Gaussian quadrature was used for the ordinary boundary elements far from the computed points but for those close to the computed points the proposed transformation was applied in the computation.

The numerical solutions of the square domain are listed in Tables 1 and 2, in which the conventional method is also employed for the boundary elements close to the computing points for the purpose of comparison. We can see that when the computed points are not too close to the boundary, both the methods are effective and obtain excellent results, but the results of the conventional method become less satisfactory as the computed points get increasingly close to the boundary, i.e., when the distance from the internal point to the boundary is equal or less than 0.01. In contrast, the results of the proposed method are steady and satisfactory even when the computed points are very close to the boundary. On the other hand, the relative errors with respect to the exact solutions are also shown in Tables 1 and 2, which further demonstrate the efficiency and the usefulness of the developed algorithm.

In addition, with the increase of the discretized boundary elements, the relative errors of the computed temperatures u and fluxes q_{x_1} related to the exact solutions at point (1E-6,3.0) on the boundary, are respectively shown in Fig. 2 and Fig. 3, from which we can observe that the convergence speeds of the computed temperatures u and fluxes q_{x_1} are still fast when the distance of the computed point to the boundary reaches 10^{-6}.

Example 2: The second example is an infinite

Table 1 Temperatures u at internal points increasingly close to the boundary

Points		Exact	Conventional (no transform)		Present	
x_1	x_2		Numerical	RE(%)	Numerical	RE(%)
0.5E-00	3.0	0.2750000E+03	0.2749770E+03	0.8348682E-02	0.2749770E+03	0.8348635E-02
0.1E-00	3.0	0.2950000E+03	0.2949620E+03	0.1288605E-01	0.2949980E+03	0.6706921E-03
0.1E-01	3.0	0.2995000E+03	0.2985807E+03	0.3069466E+00	0.2995055E+03	-0.1833155E-02
0.1E-02	3.0	0.2999500E+03	0.2987465E+03	0.4012188E+00	0.2999563E+03	-0.2116890E-02
0.1E-03	3.0	0.2999950E+03	0.2987600E+03	0.4116704E+00	0.3000014E+03	-0.2145655E-02
0.1E-04	3.0	0.2999995E+03	0.2987613E+03	0.4127259E+00	0.3000059E+03	-0.2148536E-02
0.1E-05	3.0	0.3000000E+03	0.2987615E+03	0.4128315E+00	0.3000064E+03	-0.2148824E-02
0.1E-06	3.0	0.3000000E+03	0.2987615E+03	0.4128421E+00	0.3000064E+03	-0.2148853E-02

Table 2 Fluxes q_{x_1} at internal points increasingly close to the boundary

Points		Exact	Conventional (no transform)		Present	
x_1	x_2		Numerical	RE(%)	Numerical	RE(%)
0.5E-00	3.0	-0.5E+02	-0.5004011E+02	-0.8022837E-01	-0.5004012E+02	-0.8023469E-01
0.1E-00	3.0	-0.5E+02	-0.4871553E+02	0.2568941E+01	-0.5007336E+02	-0.1467267E+00
0.1E-01	3.0	-0.5E+02	-0.2153601E+02	0.5692798E+02	-0.5009417E+02	-0.1883398E+00
0.1E-02	3.0	-0.5E+02	-0.1527870E+02	0.6944259E+02	-0.5009671E+02	-0.1934161E+00
0.1E-03	3.0	-0.5E+02	-0.1464388E+02	0.7071224E+02	-0.5009796E+02	-0.1959296E+00
0.1E-04	3.0	-0.5E+02	-0.1458039E+02	0.7083922E+02	-0.5009675E+02	-0.1934916E+00
0.1E-05	3.0	-0.5E+02	-0.1457404E+02	0.7085192E+02	-0.5008621E+02	-0.1724289E+00
0.1E-06	3.0	-0.5E+02	-0.1457340E+02	0.7085319E+02	-0.5008603E+02	-0.1720534E+00

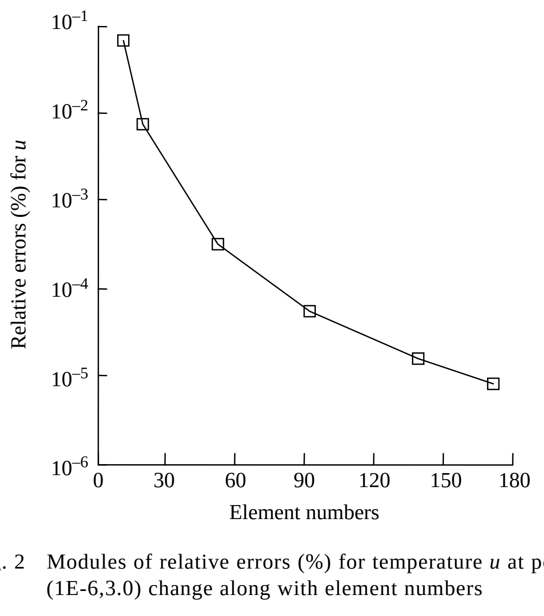

Fig. 2 Modules of relative errors (%) for temperature u at point (1E-6,3.0) change along with element numbers

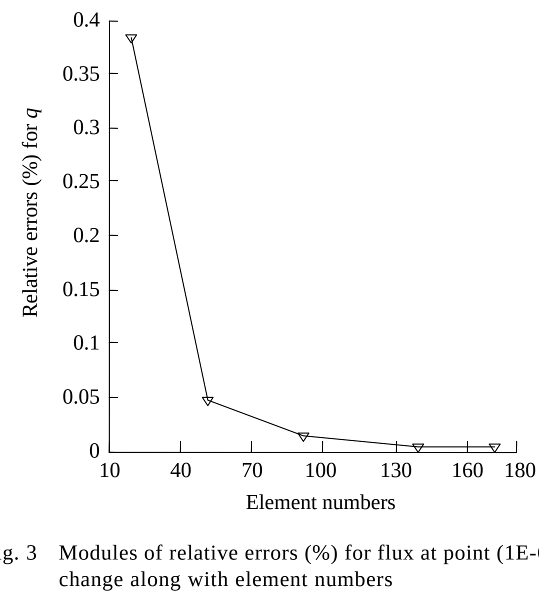

Fig. 3 Modules of relative errors (%) for flux at point (1E-6,3.0) change along with element numbers

column with radius 1.0, as shown in Fig. 4. The temperature distribution along the boundary is $u(r, \theta) = \begin{cases} 1, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$. Under steady state, the analytical solutions of the problem can be expressed by using the Fourier series (Brebbia, 1984) as in the following:

$$\frac{u(r, \theta)}{u_0} = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{n\pi}\left(\frac{r}{R}\right)^n \sin n\theta \, , \ n = 1, 3, \cdots$$

To solve this problem numerically the boundary is discretized by 24 linear elements with 48 discontinuous interpolation points. The temperatures of internal points are calculated respectively by using the conventional method and the method proposed in this paper.

The numerical results are shown in Table 3, from which it can be seen that the conventional method and the proposed method are both efficient when $r <$

Table 3 The numerical results for *u* at interior points

Points	Exact	Conventional (no transform)		Present	
$r(\theta = 37.5°)$		Numerical	RE(%)	Numerical	RE(%)
0.9000000	0.9453504E+00	0.9495592E+00	-0.4452156E+00	0.9452007E+00	0.1583243E-01
0.9600000	0.9786810E+00	0.9659999E+00	0.1295734E+01	0.9787327E+00	-0.1608756E-01
0.9900000	0.9947453E+00	0.5438051E+00	0.4533223E+02	0.9950226E+00	-0.2787484E-01
0.9990000	0.9996119E+00	0.2707488E+00	0.7291460E+02	0.9998467E+00	-0.2348945E-01
0.9999000	0.1000281E+01	0.2483153E+00	0.7517544E+02	0.1000328E+01	-0.4661747E-02
0.9999900	0.1000360E+01	0.2461421E+00	0.7539464E+02	0.1000376E+01	-0.1607714E-02
0.9999990	0.1000368E+01	0.2459255E+00	0.7541649E+02	0.1000380E+01	-0.1280416E-02
0.9999999	0.1000368E+01	0.2459038E+00	0.7541867E+02	0.1000381E+01	-0.1243939E-02

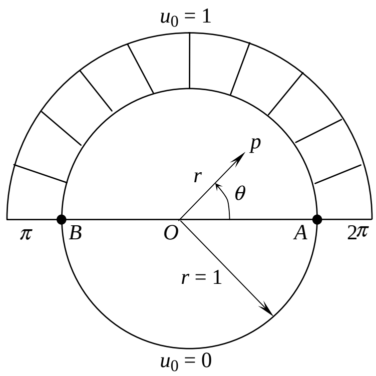

Fig. 4 Temperature boundary conditions

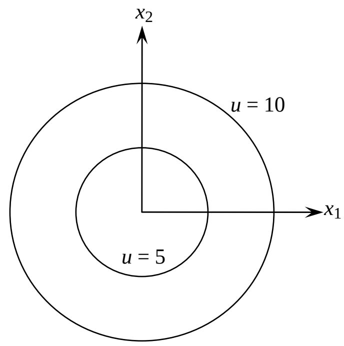

Fig. 5 Surface temperature for the cylinder

0.96, but the conventional method fails when the internal point becomes close to the boundary. However, the results obtained by using the proposed algorithm are excellently consistent with the analytical solutions even in the very unfavorable computational condition $r = 0.9999999$ or greater, i.e., when the distance of the computed point to the boundary is 10^{-7} or smaller. The relative errors are also shown in Table 3, from which we can see that the accuracy of the results of the proposed method are high and steady even when the computed points are very close to the boundary, while the relative errors of the conventional method are relatively too large to be accepted with the computed points increasingly close to the boundary. Because of the convergence problem of the above Fourier series of the analytical results, the fluxes are not given here.

Example 3: As shown in Fig. 5, this example is concern with a cylinder with infinite length whose inner and outer radii are 0.5 and 1, respectively. The corresponding boundary conditions are also described in Fig. 5.

There are 48 total arc elements divided along the boundaries of the cylinder, 18 elements over the inner boundary, and 30 elements over the outer boundary. Linear discontinuous interpolation is adopted to approximate the boundary functions. When the temperature gradients are computed, we use eight-point Gaussian integration for the ordinary integrals and sixteen-point Gaussian integration for the nearly singular integrals that occur in the computation.

The calculated results of the temperatures u and the fluxes q_{x_1} (in the x_1-direction) at internal points close to the inner boundary are shown in Tables 4 and 5, respectively. Table 4 shows that the results obtained by the conventional method are out of true with the relative errors already greater than thirteen percent when $r \leq 0.501$, i.e., when the distance from the computed point to the inner boundary is 0.001 or smaller. On the other hand, the results of the proposed algorithm are very consistent with the exact solutions with the largest error less than 0.00005% even when the distance of the internal point to the inner boundary reaches 10^{-7}. In Table 5, the fluxes computed by the conventional method are unacceptable when $r \leq 0.51$, in contrast with which the computed results of the present method are still acceptable even when the computed points are very close to the boundary.

For the internal points increasingly close to the outer boundary the computed temperatures and fluxes are shown in Fig. 6 and 8, respectively. The relative errors related to the exact solutions are shown in Figs. 7 and 9. In Fig. 7 only the errors of the proposed method are given, since the errors of the conventional method are relatively too large, and in Fig. 9 several

Table 4 Numerical results of temperature at interior points

Internal points $r(\theta = 70°)$	Exact	Conventional (no transform) Results	Conventional (no transform) RE(%)	Present Results	Present RE(%)
0.70000000	0.7427134E+01	0.7488457E+01	-0.8256649E+00	0.7427134E+01	0.1518929E-07
0.60000000	0.6315172E+01	0.6389339E+01	-0.1174423E+01	0.6315172E+01	0.5957411E-07
0.51000000	0.5142846E+01	0.5086907E+01	0.1087693E+01	0.5142846E+01	0.1152673E-05
0.50100000	0.5014413E+01	0.4347926E+01	0.1329141E+02	0.5014413E+01	-0.1460620E-04
0.50010000	0.5001443E+01	0.4214516E+01	0.1573399E+02	0.5001442E+01	0.9724757E-05
0.50001000	0.5000144E+01	0.4200390E+01	0.1599462E+02	0.5000144E+01	0.1243599E-05
0.50000100	0.5000014E+01	0.4198970E+01	0.1602084E+02	0.5000015E+01	-0.1357227E-04
0.50000010	0.5000001E+01	0.4198828E+01	0.1602347E+02	0.5000004E+01	-0.4404773E-04
0.50000001	0.5000000E+01	0.4198814E+01	0.1602373E+02	0.5000002E+01	-0.3133504E-04

Table 5 Numerical results of flux at interior points near the inner boundary

Internal points $r(\theta = 70°)$	Exact	Conventional (no transform) Results	Conventional (no transform) RE(%)	Present Results	Present RE(%)
0.55000000	0.4485734E+01	0.4485116E+01	0.1378972E-01	0.4485734E+01	-0.2458256E-06
0.51000000	0.4837557E+01	0.4223326E+01	0.1269712E+02	0.4837557E+01	-0.1954462E-06
0.50100000	0.4924459E+01	0.2701467E+01	0.4514185E+02	0.4924459E+01	-0.1041388E-04
0.50010000	0.4933321E+01	0.2490725E+01	0.4951221E+02	0.4933311E+01	0.2104592E-03
0.50001000	0.4934209E+01	0.2469512E+01	0.4995122E+02	0.4934311E+01	-0.2060567E-02
0.50000100	0.4934298E+01	0.2467390E+01	0.4999512E+02	0.4934084E+01	0.4333926E-02
0.50000010	0.4934307E+01	0.2467177E+01	0.4999951E+02	0.4933384E+01	0.1870474E-01
0.50000001	0.4934308E+01	0.2467156E+01	0.4999995E+02	0.4933914E+01	0.7969778E-02

points are outside plots for the same reason. It can be seen that the errors of the temperatures are always less than for the fluxes, although the largest error of the latter is less than 0.03%.

V. CONCLUSIONS

A new efficient transformation has been proposed to deal with the nearly singular integrals which occur in the analysis of the BEM. With the proposed transformation, the near singularities of the nearly singular integrals are removed efficiently, and fairly high accuracy of numerical results is achieved for the nearly singular integrals with standard Gauss quadrature procedures. Numerical examples of the potential problem are presented to test the proposed algorithm, with which excellent results are obtained. The results verify the feasibility and the effectiveness of the present method, and the boundary layer effect has been overcome successfully with the proposed transformation techniques in the applications. The algorithm is also general and can be applied to other problems in BEM, which will be discussed later.

Compared with existing approaches, the presented transformation is more general. The existing approaches are suitable only for cases when the regular part of the integral is constant while ours is effective whatever the regular part of the integrand may be.

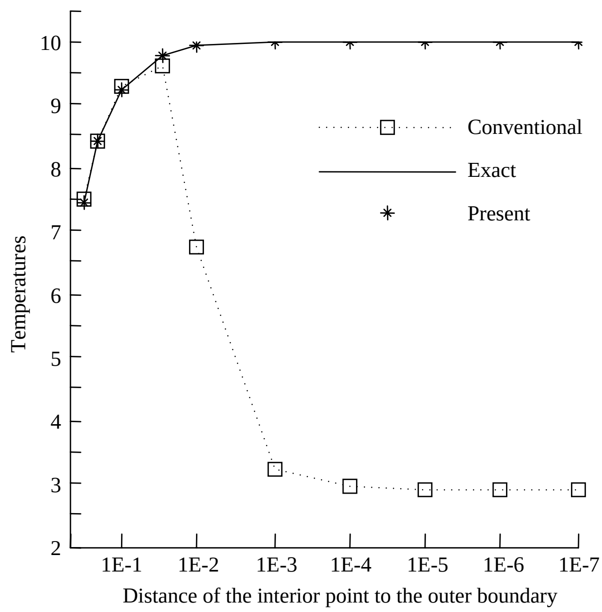

Fig. 6 Temperatures u at the interior points increasing close to the outer boundary

The remarks are given in the conclusion.

The above numerical examples show that the computation for $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ is effective and stable. Furthermore, using $\frac{\partial u}{\partial \boldsymbol{n}} = \frac{\partial u}{\partial x_1}n_1 + \frac{\partial u}{\partial x_2}n_2$, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_1}t_1 +$

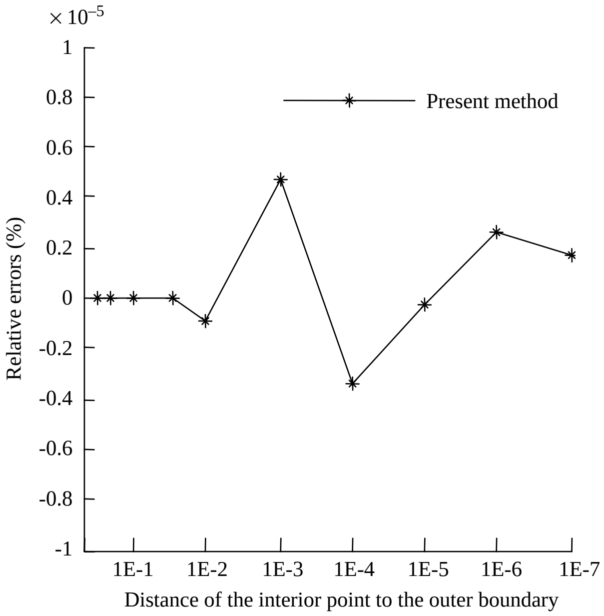

Fig. 7 Relative errors (%) of the temperature at interior points increasing close to the outer boundary

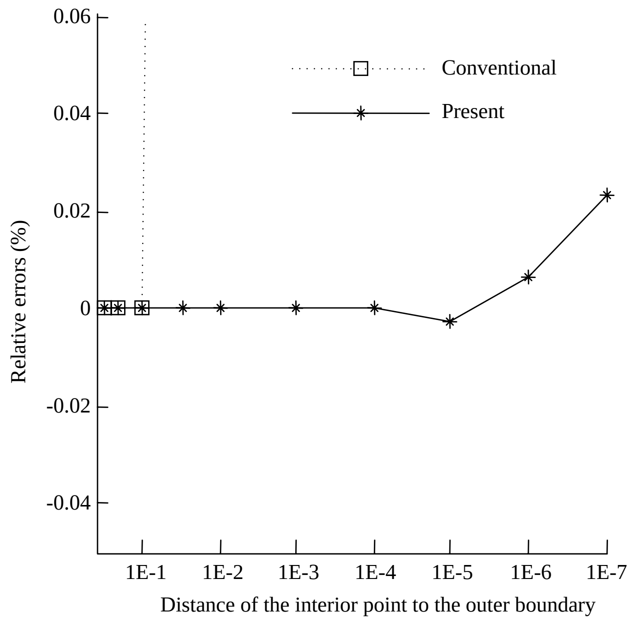

Fig. 9 Relative errors (%) of the flux q at interior points increasing close to the outer boundary

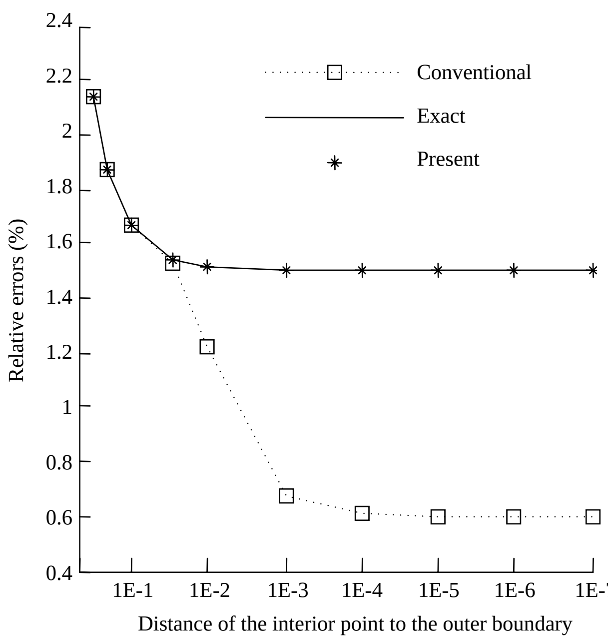

Fig. 8 Fluxes of the interior points increasing close to the outer boundary

$\frac{\partial u}{\partial x_2}t_2$, we can easily obtain normal and tangent flux and the computation is stable.

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