ANTIPLANE ELECTRO-MECHANICAL FIELD OF A PIEZOELECTRIC FINITE WEDGE UNDER A PAIR OF CONCENTRATED FORCES AND FREE CHARGES

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ABSTRACT

This paper presents general antiplane electro-mechanical field solutions for a piezoelectric finite wedge subjected to a pair of concentrated forces and free charges. The boundary conditions on the circular segment are considered as traction free and insulated. Using finite Mellin transform methods, the stress and electrical displacement in all fields of the piezoelectric finite wedge are derived analytically. Singularity orders and intensity factors of stress and electrical displacement can be obtained too. After being reduced to a problem of an antiplane edge crack or an infinite wedge in a piezoelectric medium, the results compare well with those of previous studies.

Key Words: antiplane problem, piezoelectric finite wedge, generalized intensity factor.

I. INTRODUCTION

In the past, many researchers have used Mellin transforms to solve the elastic problem of a wedge shape. Wedge problems involving piezoelectric materials are rarely reported in literature. If the piezoelectric material is polarized along the z-axis, then the wedge problem will be decoupled to inplane and antiplane problems. The antiplane field couples the antiplane elastic deformation (τ_{xz}, τ_{yz}, w) and the inplane elastic parameters (D_x, D_y, E_x, E_y). Xu and Rajapakse (2000) discussed the inplane stress singularities of piezoelectric wedges. Chue and Chen (2003) generalized Xu and Rajapakse's formulation to study the singularity orders of piezoelectric wedges under generalized plane deformation. Chen and Chue (2003) obtained the explicit forms of singular electro-mechanical field in a piezoelectric bonded wedge subjected to antiplane shear loads. Based on the complex potential function associated with eigenfunction expansion, the eigenvalue equations are also derived analytically. Chue *et al*. (2003) derived singularity orders and generalized stress, strain, electrical field and electrical displacement intensity factors in a piezoelectric wedge under antiplane deformation by using Mellin transforms.

Kargarnovin *et al*. (1997) used finite Mellin transforms to obtain the displacement and stress components in an isotropic wedge with finite radius under antiplane deformation. They drew the conclusion that the stress, τ_{rz}, and displacement, w, are divergent at the points of application of tractions. Furthermore, τ_{rz} is discontinuous on the arcs r=h_1 and r=h_2. These conclusions are incorrect. Obviously, the stress has to be continuous inside the wedge. Chue and Liu (2004) have made a comment on it. Later in 2000, Kargarnovin and Fariborz (2000) dealt with bi-material isotropic wedges with finite radius and obtained the dominant displacement field near the wedge tip.

This paper employs a finite Mellin transform method to derive, analytically, the stress and electrical displacement of all fields of a piezoelectric finite wedge. The boundary conditions on the circular segment are considered as traction free and insulated.

II. FORMULATION AND PROBLEM SOLUTION

In cylindrical coordinate system (r, θ), the constitutive equation of a piezoelectric medium polarized along the z-axis is given as:

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$$\begin{bmatrix}\tau_{\theta z}\\ \tau_{rz}\\ D_r\\ D_\theta\end{bmatrix}=\begin{bmatrix}C_{44} & 0 & 0 & -e_{15}\\ 0 & C_{44} & -e_{15} & 0\\ 0 & e_{15} & \varepsilon_{11} & 0\\ e_{15} & 0 & 0 & \varepsilon_{11}\end{bmatrix}\begin{bmatrix}\gamma_{\theta z}\\ \gamma_{rz}\\ E_r\\ E_\theta\end{bmatrix} \quad (1)$$

where τ_{ij} are the shear stresses, γ_{ij} are the shear strains, D_i are the electric displacements and E_i are the electric field vectors. The material properties C_{44}, e_{15} and ε_{11} are the elastic stiffness constant, the piezoelectric constant and the dielectric constant, respectively. The shear strain-displacement and electric field - electric potential relations are:

$$\gamma_{\theta z}=\frac{1}{r}\frac{\partial w}{\partial \theta},\quad \gamma_{rz}=\frac{\partial w}{\partial r} \quad (2)$$

and

$$E_\theta=-\frac{1}{r}\frac{\partial \phi}{\partial \theta},\quad E_r=-\frac{\partial \phi}{\partial r} \quad (3)$$

where w and ϕ are displacement and electric potential, respectively. The static equilibrium equations and Maxwell's equation under electro-static condition are given as

$$\frac{\partial}{\partial r}(r\tau_{rz})+\frac{\partial \tau_{\theta z}}{\partial \theta}=0 \quad (4)$$

$$\frac{\partial}{\partial r}(rD_r)+\frac{\partial D_\theta}{\partial \theta}=0 \quad (5)$$

In Eqs. (4) and (5), the body forces and free charges have been neglected, respectively. Substituting Eqs. (1) to (3) into Eqs. (4) and (5), the governing equations for antiplane displacement w and inplane electric potential ϕ are obtained as:

$$C_{44}\nabla^2 w+e_{15}\nabla^2\phi=0 \quad (6)$$

$$e_{15}\nabla^2 w-\varepsilon_{11}\nabla^2\phi=0 \quad (7)$$

where ∇^2 is the Laplacian in (r, θ). The solutions of Eqs. (6) and (7) are:

$$\nabla^2 w=0 \quad (8)$$

$$\nabla^2\phi=0 \quad (9)$$

The finite Mellin transform on a function $W(r, \theta)$ of the second kind is defined as (Sneddon, 1972)

$$m_2[W(r,\theta);S]\equiv W^*(S,\theta)=\int_0^a\left(\frac{a^{2S}}{r^{S+1}}+r^{S-1}\right)W(r,\theta)dr \quad (10)$$

The inversion formula of this transform is in the form (Kargarnovin *et al*., 1997)

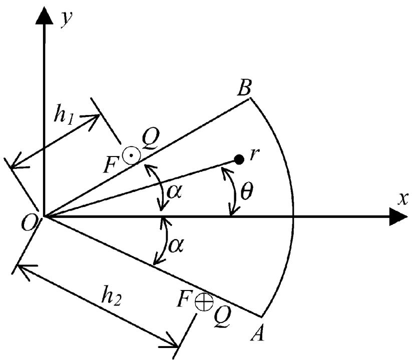

Fig.1 A piezoelectric wedge with a wedge angle, 2α, and, a, finite radius a subjected to a pair of concentrated forces, F, and free charges, Q

$$m_2^{-1}[W^*(S,\theta);r]\equiv W(r,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}W^*(S,\theta)dS \quad (11)$$

Figure 1 shows a piezoelectric finite wedge with a wedge angle 2α and a finite radius a. The radial edges ($\theta=\pm\alpha$) are subjected to a pair of concentrated forces F and free charges Q:

$$\tau_{\theta z}(r,\alpha)=F\delta(r-h_1)$$

$$\tau_{\theta z}(r,-\alpha)=F\delta(r-h_2)$$

$$D_\theta(r,\alpha)=Q\delta(r-h_1)$$

$$D_\theta(r,-\alpha)=Q\delta(r-h_2) \quad (12)$$

where $h_1\le h_2$. The boundary conditions of the circular segment of the wedge ($r=a$) are assumed to be traction-free and electrically open:

$$\tau_{rz}(a,\theta)=0 \quad (13)$$

$$D_r(a,\theta)=0 \quad (14)$$

According to Eqs. (1) to (3), Eqs. (13) and (14) can be rewritten as:

$$\frac{\partial w}{\partial r}(a,\theta)=0 \quad (15)$$

and

$$\frac{\partial \phi}{\partial r}(a,\theta)=0 \quad (16)$$

Because of condition (15), we apply the finite Mellin transform of the second kind to (8). The result gives

$$\frac{\partial^2 w^*}{\partial \theta^2}+S^2 w^*=0 \quad (17)$$

provided that

$$[(a^{2S}r^{-S+1} + r^{S+1})\frac{\partial w}{\partial r} + (a^{2S}r^{-S} - r^{S})Sw]_{r\to 0} = 0 \tag{18}$$

The solution of (17) is

$$w^* = A(S)\cos S\theta + B(S)\sin S\theta \tag{19}$$

where A and B are unknown functions of S. Using the same method, the solution of transformed electric potential ϕ^* is

$$\phi^* = C(S)\cos S\theta + D(S)\sin S\theta \tag{20}$$

where C and D are unknown functions of S also. The functions A, B, C and D can be determined by boundary conditions. Substituting Eqs. (1) to (3) into boundary conditions (12) and applying the finite Mellin transform of the second kind, the results are

$$C_{44}(-AS\sin S\alpha + BS\cos S\alpha) + e_{15}(-C\sin S\alpha + D\cos S\alpha)$$

$$= F(a^{2S}h_1^{-S} + h_1^S) \tag{21}$$

$$e_{15}(-AS\sin S\alpha + BS\cos S\alpha) - \varepsilon_{11}(-CS\sin S\alpha + D\cos S\alpha)$$

$$= Q(a^{2S}h_1^{-S} + h_1^S) \tag{22}$$

for $\theta = \alpha$, and

$$C_{44}(AS\sin S\alpha + BS\cos S\alpha) + e_{15}(CS\sin S\alpha + D\cos S\alpha)$$

$$= F(a^{2S}h_2^{-S} + h_2^S) \tag{23}$$

$$e_{15}(AS\sin S\alpha + BS\cos S\alpha) - \varepsilon_{11}(CS\sin S\alpha + D\cos S\alpha)$$

$$= Q(a^{2S}h_2^{-S} + h_2^S) \tag{24}$$

for $\theta = -\alpha$. The solutions of Eqs. (21) to (24) are

$$A = \frac{(\varepsilon_{11}F + e_{15}Q)(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S)}{(C_{44}\varepsilon_{11} + e_{15}^2)2S\sin S\alpha} \tag{25}$$

$$B = \frac{(\varepsilon_{11}F + e_{15}Q)(a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)}{(C_{44}\varepsilon_{11} + e_{15}^2)2S\cos S\alpha} \tag{26}$$

$$C = \frac{(e_{15}F - C_{44}Q)(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S)}{(C_{44}\varepsilon_{11} + e_{15}^2)2S\sin S\alpha} \tag{27}$$

$$D = \frac{(e_{15}F - C_{44}Q)(a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)}{(C_{44}\varepsilon_{11} + e_{15}^2)2S\cos S\alpha} \tag{28}$$

From Eqs. (19), (20), (25) to (28), the transformed antiplane displacement and electric potential becomes

$$w^*(S,\theta) = \frac{(\varepsilon_{11}F + e_{15}Q)(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S)}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\frac{\cos S\theta}{S\sin S\alpha} + \frac{(\varepsilon_{11}F + e_{15}Q)(a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\frac{\sin S\theta}{S\cos S\alpha} \tag{29}$$

$$\phi^*(S,\theta) = \frac{(e_{15}F - C_{44}Q)(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S)}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\frac{\cos S\theta}{S\sin S\alpha} + \frac{(e_{15}F - C_{44}Q)(a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\frac{\sin S\theta}{S\cos S\alpha} \tag{30}$$

By using the inverse transform (11), the antiplane displacement w and the electric potential ϕ are obtained

$$w(r,\theta) = \frac{\varepsilon_{11}F + e_{15}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}[(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S) \cdot \frac{\cos S\theta}{S\sin S\alpha} + (a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)\frac{\sin S\theta}{S\cos S\alpha}]dS \tag{31}$$

$$\phi(r,\theta) = \frac{e_{15}F - C_{44}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}[(a^{2S}h_2^{-S} + h_2^S - a^{2S}h_1^{-S} - h_1^S) \cdot \frac{\cos S\theta}{S\sin S\alpha} + (a^{2S}h_2^{-S} + h_2^S + a^{2S}h_1^{-S} + h_1^S)\frac{\sin S\theta}{S\cos S\alpha}]dS \tag{32}$$

The residual theorem is used to carry out the inverse integral. From condition (18) the strip of integration $Re[s] = c$ is defined as

$$-S_n < c < S_n \tag{33}$$

where S_n are the poles of integrands in Eqs. (31) and (32). Three regions $r \le h_1$, $h_1 \le r \le h_2$, and $a \ge r \ge h_2$ are considered separately in the integration. The results are:

$$w(r,\theta) = \frac{\varepsilon_{11}F + e_{15}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{r}{h_2})^{\frac{k\pi}{\alpha}} - (\frac{r}{h_1})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha})\} \tag{34}$$

$$\phi(r,\theta) = \frac{e_{15}F - C_{44}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{r}{h_2})^{\frac{k\pi}{\alpha}} - (\frac{r}{h_1})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha})\} \tag{35}$$

for $r \le h_1$,

$$w(r,\theta) = \frac{\varepsilon_{11}F + e_{15}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{r}{h_2})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha}) + \frac{\ln h_1 - \ln r}{\alpha}\} \tag{36}$$

$$\phi(r,\theta) = \frac{e_{15}F - C_{44}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{r}{h_2})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha}) + \frac{\ln h_1 - \ln r}{\alpha}\} \tag{37}$$

for $h_1 \le r \le h_2$, and

$$w(r,\theta) = \frac{\varepsilon_{11}F + e_{15}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha}) + \frac{\ln h_1 - \ln h_2}{\alpha}\} \tag{38}$$

$$\phi(r,\theta) = \frac{e_{15}F - C_{44}Q}{2(C_{44}\varepsilon_{11} + e_{15}^2)}\{\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}} - (\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha}) + \sum_{k=0}^{\infty}(-1)^k\frac{2}{(2k+1)\pi}[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}} + (\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot \sin(\frac{(2k+1)\pi\theta}{2\alpha}) + \frac{\ln h_1 - \ln h_2}{\alpha}\} \tag{39}$$

for $r \ge h_2$. The stresses τ_{rz}, $\tau_{\theta z}$ and electric displacement D_r, D_θ, except on the circular arcs $r=h_1$ and $r=h_2$, are then obtained as:

$$\begin{aligned}
&\tau_{rz}(r,\theta)\\
&=\frac{F}{2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{k\pi}{\alpha}-1}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}})\\
&\quad-\frac{1}{h_2}(\frac{r}{h_2})^{\frac{k\pi}{\alpha}-1}+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{k\pi}{\alpha}-1}]\cos(\frac{k\pi\theta}{\alpha})\\
&\quad+\sum_{k=0}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}-1}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}\\
&\quad+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}-1}\\
&\quad+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}-1}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})\}
\end{aligned}\tag{40}$$

$$\begin{aligned}
&\tau_{\theta z}(r,\theta)\\
&=\frac{F}{2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{k\pi}{\alpha}-1}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}})\\
&\quad+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{k\pi}{\alpha}-1}-\frac{1}{h_1}(\frac{r}{h_1})^{\frac{k\pi}{\alpha}-1}]\sin(\frac{k\pi\theta}{\alpha})\\
&\quad+\sum_{k=0}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}-1}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}\\
&\quad+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}-1}\\
&\quad+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}-1}]\cdot\cos(\frac{(2k+1)\pi\theta}{2\alpha})\}
\end{aligned}\tag{41}$$

$$\begin{aligned}
&D_r(r,\theta)\\
&=\frac{Q}{2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{k\pi}{\alpha}-1}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}})\\
&\quad-\frac{1}{h_2}(\frac{r}{h_2})^{\frac{k\pi}{\alpha}-1}+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{k\pi}{\alpha}-1}]\cos(\frac{k\pi\theta}{\alpha})\\
&\quad+\sum_{k=0}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}-1}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}\\
&\quad+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}-1}\\
&\quad+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}-1}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})\}
\end{aligned}\tag{42}$$

$$\begin{aligned}
&D_\theta(r,\theta)\\
&=\frac{Q}{2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{k\pi}{\alpha}-1}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}})\\
&\quad+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{k\pi}{\alpha}-1}-\frac{1}{h_1}(\frac{r}{h_1})^{\frac{k\pi}{\alpha}-1}]\sin(\frac{k\pi\theta}{\alpha})\\
&\quad+\sum_{k=0}^{\infty}(-1)^k[\frac{1}{a}(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}-1}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}\\
&\quad+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})+\frac{1}{h_2}(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}-1}\\
&\quad+\frac{1}{h_1}(\frac{r}{h_1})^{\frac{(2k+1)\pi}{2\alpha}-1}]\cos(\frac{(2k+1)\pi\theta}{2\alpha})\}
\end{aligned}\tag{43}$$

for $r<h_1$,

$$\begin{aligned}
&\tau_{rz}(r,\theta)\\
&=\frac{F}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}})\\
&\quad-(\frac{r}{h_2})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k\\
&\quad\cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})\\
&\quad+(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}-(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})-1\}
\end{aligned}\tag{44}$$

$$\begin{aligned}
&\tau_{\theta z}(r,\theta)\\
&=\frac{F}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}})\\
&\quad+(\frac{r}{h_2})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k\\
&\quad\cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})\\
&\quad+(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\cos(\frac{(2k+1)\pi\theta}{2\alpha})\}
\end{aligned}\tag{45}$$

$$\begin{aligned}
&D_r(r,\theta)\\
&=\frac{Q}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}})\\
&\quad-(\frac{r}{h_2})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k\\
&\quad[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}})\\
&\quad+(\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}-(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})-1\}
\end{aligned}\tag{46}$$

$$D_\theta(r,\theta) = \frac{Q}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{r}{h_2})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k \cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{r}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\cos(\frac{(2k+1)\pi\theta}{2\alpha})\} \quad (47)$$

for $h_1<r<h_2$, and

$$\tau_{rz}(r,\theta) = \frac{F}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k \cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) - (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}}-(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})\} \quad (48)$$

$$\tau_{\theta z}(r,\theta) = \frac{F}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k \cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\cos(\frac{(2k+1)\pi\theta}{2\alpha})\} \quad (49)$$

$$D_r(r,\theta) = \frac{Q}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k \cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) - (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}}-(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\sin(\frac{(2k+1)\pi\theta}{2\alpha})\} \quad (50)$$

$$D_\theta(r,\theta) = \frac{Q}{2r\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{r}{a})^{\frac{k\pi}{\alpha}}((\frac{h_2}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}) + (\frac{h_2}{r})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{r})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k \cdot[(\frac{r}{a})^{\frac{(2k+1)\pi}{2\alpha}}((\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}) + (\frac{h_2}{r})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{r})^{\frac{(2k+1)\pi}{2\alpha}}]\cos(\frac{(2k+1)\pi\theta}{2\alpha})\} \quad (51)$$

for $a \geq r > h_2$.

The singularity order for stresses and electrical displacements is $(\pi/2\alpha)-1$ when $-1<(\pi/2\alpha)-1<0$. The singularity order is independent of the wedge radius a and coincides with the result of Chue *et al*. (2003) for infinite wedge problem. No singularities are observed for $\alpha\leq\pi/2$. In addition, the order becomes a conventional square root when the wedge structure becomes a crack in a piezoelectric medium, i.e. $\alpha=\pi$.

The stresses and electric displacements on the circular arcs $r=h_1$ and $r=h_2$ have to be dealt with separately. For example, τ_{rz} and D_r can be derived by using the following equations:

$$\tau_{rz}(r,\theta) = -\frac{C_{44}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}Sw^*(S,\theta)dS - \frac{e_{15}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}S\phi^*(S,\theta)dS \quad (52)$$

$$D_r(r,\theta) = -\frac{e_{15}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}Sw^*(S,\theta)dS + \frac{\varepsilon_{11}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-S}S\phi^*(S,\theta)dS \quad (53)$$

After substituting w^* and ϕ^* (Eqs. (19) and (20)) into the above two equations and letting $r=h_1$ and $r=h_2$, we carry out a lengthy calculation. The results are:

$$\tau_{rz}(h_1,\theta) = \frac{F}{2h_1\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}+(\frac{h_1}{a})^{\frac{2k\pi}{\alpha}} - (\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\sin(\frac{(2k+1)\pi\theta}{2\alpha})-\frac{1}{2}\} \quad (54)$$

$$D_r(h_1,\theta) = \frac{Q}{2h_1\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}+(\frac{h_1}{a})^{\frac{2k\pi}{\alpha}} -(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\cdot\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\sin(\frac{(2k+1)\pi\theta}{2\alpha})-\frac{1}{2}\} \quad (55)$$

for $r=h_1$, and

$$\tau_{rz}(h_2,\theta) = \frac{F}{2h_2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{2k\pi}{\alpha}} -(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}}-(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\sin(\frac{(2k+1)\pi\theta}{2\alpha})-\frac{1}{2}\} \quad (56)$$

$$D_r(h_2,\theta) = \frac{Q}{2h_2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_2}{a})^{\frac{2k\pi}{\alpha}} -(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\cos(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_2}{a})^{\frac{(2k+1)\pi}{\alpha}}-(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\sin(\frac{(2k+1)\pi\theta}{2\alpha})-\frac{1}{2}\} \quad (57)$$

for $r=h_2$.

Substituting $r=h_1$ and $r=h_2$ in Eqs. (45) and (47) the stress $\tau_{\theta z}$ and electrical displacement D_θ become:

$$\tau_{\theta z}(h_1,\theta) = \frac{F}{2h_1\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{2k\pi}{\alpha}} +(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\cos(\frac{(2k+1)\pi\theta}{2\alpha})+\frac{\cos(\frac{\pi\theta}{4\alpha})+\sin(\frac{\pi\theta}{4\alpha})}{2\cos(\frac{\pi\theta}{4\alpha})-2\sin(\frac{\pi\theta}{4\alpha})}\},\quad \theta\neq\pm\alpha \quad (58)$$

$$D_\theta(h_1,\theta) = \frac{Q}{2h_1\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}-(\frac{h_1}{a})^{\frac{2k\pi}{\alpha}} +(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\cos(\frac{(2k+1)\pi\theta}{2\alpha})+\frac{\cos(\frac{\pi\theta}{4\alpha})+\sin(\frac{\pi\theta}{4\alpha})}{2\cos(\frac{\pi\theta}{4\alpha})-2\sin(\frac{\pi\theta}{4\alpha})}\},\quad \theta\neq\pm\alpha \quad (59)$$

for $r=h_1$, and

$$D_\theta(h_2,\theta) = \frac{F}{2h_2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}+(\frac{h_2}{a})^{\frac{2k\pi}{\alpha}} -(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_1}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\cos(\frac{(2k+1)\pi\theta}{2\alpha})+\frac{\cos(\frac{\pi\theta}{4\alpha})-\sin(\frac{\pi\theta}{4\alpha})}{2\cos(\frac{\pi\theta}{4\alpha})+2\sin(\frac{\pi\theta}{4\alpha})}\},\quad \theta\neq\pm\alpha \quad (60)$$

$$D_\theta(h_2,\theta) = \frac{Q}{2h_2\alpha}\cdot\{\sum_{k=1}^{\infty}(-1)^k[-(\frac{h_2}{a})^{\frac{k\pi}{\alpha}}(\frac{h_1}{a})^{\frac{k\pi}{\alpha}}+(\frac{h_2}{a})^{\frac{2k\pi}{\alpha}} -(\frac{h_1}{h_2})^{\frac{k\pi}{\alpha}}]\sin(\frac{k\pi\theta}{\alpha})+\sum_{k=0}^{\infty}(-1)^k[(\frac{h_2}{a})^{\frac{(2k+1)\pi}{2\alpha}} \cdot(\frac{h_1}{a})^{\frac{(2k+1)\pi}{2\alpha}}+(\frac{h_2}{a})^{\frac{(2k+1)\pi}{\alpha}}+(\frac{h_1}{h_2})^{\frac{(2k+1)\pi}{2\alpha}}] \cdot\cos(\frac{(2k+1)\pi\theta}{2\alpha})+\frac{\cos(\frac{\pi\theta}{4\alpha})-\sin(\frac{\pi\theta}{4\alpha})}{2\cos(\frac{\pi\theta}{4\alpha})+2\sin(\frac{\pi\theta}{4\alpha})}\},\quad \theta\neq\pm\alpha \quad (61)$$

for $r=h_2$. In deriving Eqs. (58) to (61), the following identities have been used:

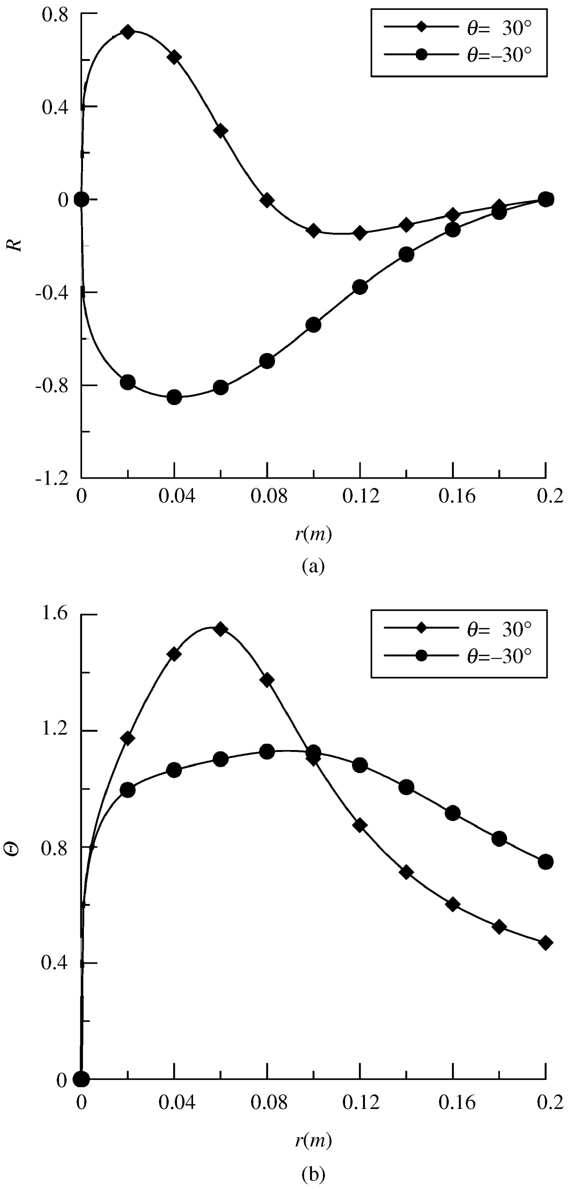

Fig.2 Normalized stress and electrical displacement distributions when a=0.2m, h_1=0.08m, h_2=0.14m, and 2α=150° (a) $R=\tau_{rz}(a/F)=D_r(a/Q)$, (b) $\Theta=\tau_{\theta z}(a/F)=D_\theta(a/Q)$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k\pi}\cos\left(\frac{k\pi\theta}{\alpha}\right)$$
$$=\frac{1}{\pi}\mathrm{Re}\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}\cdot e^{i\frac{k\pi\theta}{\alpha}}\right]$$
$$=\frac{1}{\pi}\mathrm{Re}\left[\ln\left[e^{i\frac{\pi\theta}{\alpha}}+1\right]\right],\ \theta\neq\pm\alpha \tag{62}$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{2}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi\theta}{2\alpha}\right)$$
$$=\frac{1}{\pi}\mathrm{Im}\left[\sum_{k=0}^{\infty}(-1)^{k}\frac{2}{(2k+1)}\cdot e^{i\frac{(2k+1)\pi\theta}{2\alpha}}\right]$$
$$=\frac{2}{\pi}\mathrm{Im}\left[\tan^{-1}\left(e^{i\frac{\pi\theta}{2\alpha}}\right)\right],\ \theta\neq\pm\alpha \tag{63}$$

$$\ln[x+1]=x\sum_{k=0}^{\infty}\frac{(-x)^k}{k+1},\ -1<x\le 1 \tag{64}$$

$$\tan^{-1}x=x\cdot\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2k+1},\ |x|<1 \tag{65}$$

The identities of (64) and (65) are from the table of the Handbook (Tuma, 1979). These two identities are restricted for $\theta\neq\pm\alpha$. Consequently, the quantities $\tau_{\theta z}$ and D_θ in Eqs. (58)~(61) defined on the circular arcs $r=h_1$ and $r=h_2$ are invalid for $\theta=\pm\alpha$.

We define $R(r, \theta)=\tau_{rz}(a/F)=D_r(a/Q)$ and $\Theta(r, \theta)=\tau_{\theta z}(a/F)=D_\theta(a/Q)$ as the normalized stresses and electrical displacements, respectively. When a=0.2m, h_1=0.08m, h_2=0.14m, and 2α=150°, Figs. 2(a) and 2(b) plot the distributions of R and Θ along $\theta=\pm30°$, respectively. Since the singularities disappear for $2\alpha\le180°$, the stresses and electrical displacements remain in finite values. It can be also seen that R and Θ are continuous across the circular arcs $r=h_1$ and $r=h_2$. In addition, R (i.e. τ_{rz} and D_r) vanishes along circular edge $r=a$ according to boundary conditions.

The results of R and Θ for case α=270° are plotted in Fig. 3. It appears that the stresses and displacements approach infinity as $r\to0$.

Physically speaking, $\tau_{\theta z}$ and D_θ are zero at the points $(h_1, -\alpha)$ and (h_2, α). In addition, $\tau_{\theta z}$ and D_θ will diverge at the points (h_1, α) and $(h_2, -\alpha)$ where the concentrated shear forces and free charges are applied. These phenomena can be easily observed if we substitute Eqs. (29) and (30) into the following equations:

$$\tau_{\theta z}(r,\theta)=\frac{C_{44}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}r^{-S}\frac{\partial}{\partial\theta}w^*(S,\theta)dS$$
$$+\frac{e_{15}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}r^{-S}\frac{\partial}{\partial\theta}\phi^*(S,\theta)dS$$
$$D_\theta(r,\theta)=\frac{e_{15}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}r^{-S}\frac{\partial}{\partial\theta}w^*(S,\theta)dS$$
$$-\frac{\varepsilon_{11}}{r}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}r^{-S}\frac{\partial}{\partial\theta}\phi^*(S,\theta)dS \tag{66}$$

III. STRESS AND ELECTRICAL DISPLACEMENT INTENSITY FACTORS

Similar to the concept of conventional fracture mechanics, the distributions of stress and electrical displacement near the singular point of the wedge can be written as

$$\tau_{\theta z}(r,\theta)=K_{III}^{\tau}\cdot r^{\lambda-1}\cdot f_{\theta z}(\theta) \tag{67}$$

$$\tau_{rz}(r,\theta)=K_{III}^{\tau}\cdot r^{\lambda-1}\cdot f_{rz}(\theta) \tag{68}$$

$$D_\theta(r,\theta)=K_{III}^{D}\cdot r^{\lambda-1}\cdot g_\theta(\theta) \tag{69}$$

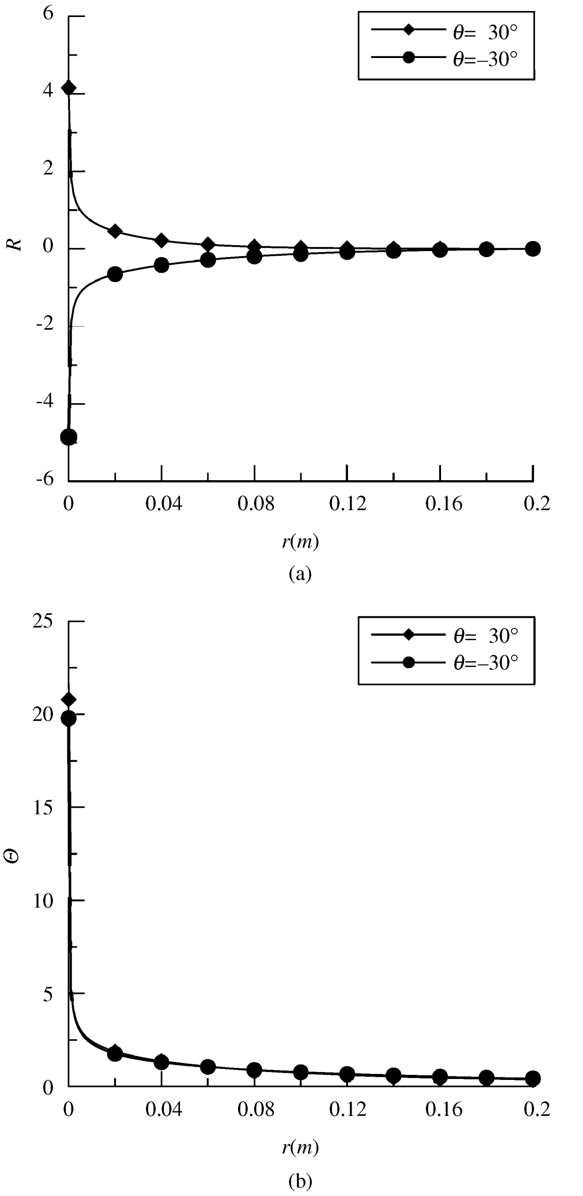

Fig. 3 Normalized stress and electrical displacement distributions when a=0.2m, h_1=0.08m, h_2=0.14m, and 2α= 270° (a) $R=\tau_{rz}(a/F)=D_r(a/Q)$, (b) $\Theta=\tau_{\theta z}(a/F)=D_\theta(a/Q)$

$$D_r(r, \theta) = K_{III}^D \cdot r^{\lambda-1} \cdot g_r(\theta) \tag{70}$$

where Re[λ–1] is the singularity order, K_{III}^τ the generalized stress intensity factor, K_{III}^D the generalized electrical displacement intensity factor, and $f_{ij}(\theta)$ and $g_i(\theta)$ are the angular functions. Higher order terms in Eqs. (67) to (70) have been discarded.

Comparing Eqs. (67) to (70) with Eqs. (40) to (43), we obtain the intensity factors and the associated angular functions:

$$K_{III}^\tau = \frac{\sqrt{2\pi}F}{2\alpha} \cdot [(\frac{h_2}{a^2})^{\frac{\pi}{2\alpha}} + (\frac{h_1}{a^2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_1})^{\frac{\pi}{2\alpha}}] \tag{71}$$

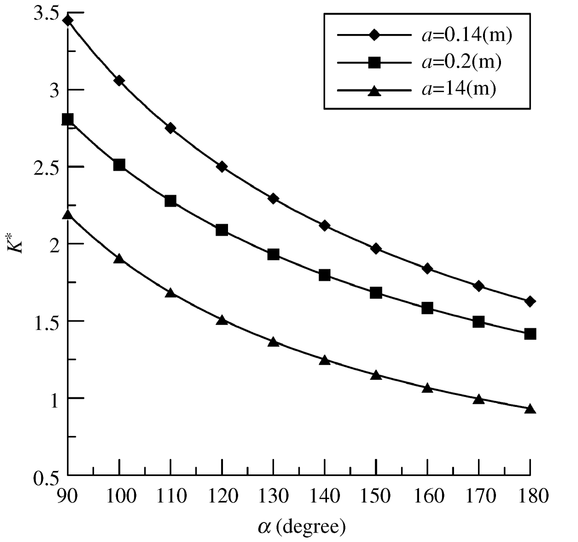

Fig. 4 Variations of normalized stress and electrical displacement intensity factors with half wedge angle at different finite radius a (h_1=0.08m, h_2=0.14m)

$$K_{III}^D = \frac{\sqrt{2\pi}Q}{2\alpha} \cdot [(\frac{h_2}{a^2})^{\frac{\pi}{2\alpha}} + (\frac{h_1}{a^2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_1})^{\frac{\pi}{2\alpha}}] \tag{72}$$

$$f_{\theta z}(\theta) = g_\theta(\theta) = \frac{1}{\sqrt{2\pi}}\cos(\frac{\pi\theta}{2\alpha}) \tag{73}$$

$$f_{rz}(\theta) = g_r(\theta) = \frac{1}{\sqrt{2\pi}}\sin(\frac{\pi\theta}{2\alpha}) \tag{74}$$

According to the expression forms in Eqs. (71) and (72), a normalized intensity factor K^* is defined as follows:

$$K^* = K_{III}^\tau h_2^{\pi/2\alpha}/F = K_{III}^D h_2^{\pi/2\alpha}/Q$$
$$= \sqrt{\frac{\pi}{2}}\frac{1}{\alpha}[(\frac{h_2}{a})^{\frac{\pi}{\alpha}} + (\frac{h_2 h_1}{a^2})^{\frac{\pi}{2\alpha}} + 1 + (\frac{h_2}{h_1})^{\frac{\pi}{2\alpha}}] \tag{75}$$

Fig. 4 plots the variations of normalized intensity factor K^* with half wedge angle at different finite radii, a, when h_1=0.08m, h_2=0.14m. It shows that a finite wedge with smaller wedge angle, 2α, and smaller radius, a, results in larger generalized intensity factor. Physically speaking, the fracture toughness of a piezoelectric material depends on the wedge angle. In general, it needs more energy to fracture a wedge with a smaller wedge angle under in-plane deformation (Chen, *et al.*, 1996; Dunn, *et al.*, 1997). To the authors' knowledge, no suitable criterion has existed to evaluate the fracture behavior of a wedge structure till now. Therefore, the values of generalized stress and electrical displacement intensity factors are temporarily meaningless for predicting the fracture possibility except for the case of a crack in a medium (2α=360°).

For an infinite piezoelectric wedge (i.e. $a\to\infty$), the generalized intensity factors for stress and electrical displacement becomes:

$$K_{III}^{\tau} = \frac{\sqrt{2\pi}F}{2\alpha}\cdot[(\frac{1}{h_2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_1})^{\frac{\pi}{2\alpha}}] \tag{76}$$

$$K_{III}^{D} = \frac{\sqrt{2\pi}Q}{2\alpha}\cdot[(\frac{1}{h_2})^{\frac{\pi}{2\alpha}} + (\frac{1}{h_1})^{\frac{\pi}{2\alpha}}] \tag{77}$$

Furthermore, if we let $h=h_1=h_2$, the results are:

$$K_{III}^{\tau} = \frac{F}{\alpha}\sqrt{\frac{2\pi}{h^{\pi/\alpha}}} \tag{78}$$

$$K_{III}^{D} = \frac{Q}{\alpha}\sqrt{\frac{2\pi}{h^{\pi/\alpha}}} \tag{79}$$

which compare well with those of Chue *et al*. (2003).

IV. CONCLUSION

The antiplane electro-mechanical fields of a piezoelectric finite wedge under a pair of concentrated forces and free charges have been obtained in this paper. The results show that the stresses and electrical displacements, with or without singularities, are continuous. In addition, the generalized stress and electrical displacement intensity factors for finite a or $a\to\infty$ have been derived. The results of the case when $a\to\infty$ are compared well with those of previous studies.

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